GRANT/AMES

SEPTEMBER 1986

LIDS-TH-1609

233p.

Research Supported By:

NASA Ames and Langley Research Centers
Grant NASA/NAG2-297

IN-34261

A METHODOLOGY FOR DESIGNING ROBUST MULTIVARIABLE NONLINEAR CONTROL SYSTEMS

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This report is based on the unaltered thesis of Daniel Backer Grunberg submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the Massachusetts Institute of Technology in August 1986. The research was conducted at the M.I.T. Laboratory for Information and Decision Systems with support provided by NASA Ames and Langley Research Centers under grant NASA/NAG2-297.

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S.B. Electrical Engineering, M.I.T.
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Submitted to the Department of

ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August 1986

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Submitted to the Department of Electrical Engineering and Computer Science on August 8, 1986 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

A new methodology is described for the design of nonlinear dynamic controllers for nonlinear multivariable systems providing guarantees of closed-loop stability, performance, and robustness. The methodology is an extension of the Linear-Quadratic-Gaussian with Loop-Transfer-Recovery (LQG/LTR) methodology for linear systems, thus hinging upon the idea of constructing an approximate inverse operator for the plant. A major feature of the methodology is a unification of both the state-space and input-output formulations.

In addition, new results on stability theory, nonlinear state estimation, and optimal nonlinear regulator theory are presented, including the guaranteed global properties of the extended Kalman filter and optimal nonlinear regulators.

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ACKNOWLEDGEMENTS

I am grateful to Professor Michael Athans for his invaluable insights, guidance, and support during the course of this research. Without his drive and enthusiasm for research I would not have started this project and would certainly never have brought it to its present form. His contribution to my education goes far beyond his technical guidance of this thesis project.

I would also like to thank the remaining members of my thesis committee for their many helpful comments and suggestions. Specifically, I thank Professor Sanjoy K. Mitter, Professor John N. Tsitsiklis, and Doctor Gunter Stein for their contributions to this research. Doctors H. Austin Spang, John F. Cassidy, and David Mercaldi deserve thanks for giving me my start in systems and control theory in the first place, years ago.

In addition, I would like to thank some of my fellow students for listening, arguing, and just plain making life at LIDS more pleasant. They include Petros Kapasouris, Richard LaMaire, David Milich, Jason Papastavrou, Tony Rodriguez, Jeff Shamma, Wilma Quinn, and Jim Walton. Thanks are also due to Doug Chin, John Iler, Rob Schaefer, and Steve Yao, who made me realize that arguments could be about things other than control theory. I and many of my fellow students are grateful for the assistance of Fifa Monserrate in typing papers and other details of student life at LIDS. I also thank my brother Noah for the wonderful job he did drawing the figures for this thesis.

I would like to thank my family for their active support of my many activities. I would also like to thank one very important member of my family-to-be, Elaine, for her love and patience during the course of this research, for making me realize that not everything had to be an argument, and for the time we spent together away from our respective schoolwork.

This research was conducted at the M.I.T. Laboratory for Information and Decision Systems with support provided by a National Science Foundation Graduate Fellowship, by the NASA Ames and Langley Research Centers under grant NASA/NAG2-297, and by a grant from the General Electric Company.

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CHAPTER 1. INTRODUCTION

1.1 Overview

1.1.1 Motivation

The feedback control systems that we build and operate today are exceedingly complex and nonlinear in their operation. Even our best efforts at modeling these systems can produce only low order nonlinear models that describe the behavior of the system over some limited operating regime. Frequently, we often generate even simpler, linear models of these systems, valid in some even smaller operating regime. We then apply methods that have been developed for controlling linear systems. Engineers succeed at applying these methods in the real world because frequently the region of desired operation is within a region of approximate linearity. Even if this is not the case, we can sometimes alter specifications to shrink the region of operation; virtually any system will behave almost linearly if not pushed too hard. We are then justified in modeling a real world system by a linear one and applying linear theories and design methodologies to it.

However, as we build faster and more complex dynamic feedback systems (e.g. aircraft, robots, reactors, etc.), we find that this method does not suffice. As we begin to demand higher performance from our systems, we begin to require something more than linear models and theories. We begin to require a nonlinear control system design theory that we can apply to the nonlinear models we must derive.

Current approaches to this problem are ad hoc in nature: the problem is broken down into several linear ones, with linear theory used on each piece. Then all the linear controllers are combined into one global nonlinear controller for the original system. Unfortunately, this method, called *gain-scheduling*, comes with no guarantees. There is currently no sound theoretical basis for gain-scheduling, and while there have been some successes, problems have been reported in high performance designs. Generally, it has been successful when the system is not pushed too hard, i.e. changes regimes slowly.

This thesis describes an approach without these drawbacks. It describes a methodology for the control of nonlinear systems in which a nonlinear dynamic compensator is designed directly, with no intermediate linearization and combination steps. It has turned out that, in addition to being a more aesthetically appealing approach, this technique guarantees several desirable properties for the nonlinear closed-loop system on a sound mathematical basis.

1.1.2 Contributions of the Thesis

The approach presented here is an extension of the Linear-Quadratic-Gaussian with Loop Transfer Recovery (LQG/LTR) methodology for linear systems [1,2,3], which has been recently developed. We call the extension the Nonlinear-Model-Based-Compensator with Loop-Operator-Recovery (NMBC/LOR) methodology.

Fortunately, many of the results in the linear case can be extended to the nonlinear case with little or no conceptual change. The methodology outlined here is hoped to be the start of a complete prescription for designing control systems for nonlinear systems, providing

(a) guaranteed closed-loop stability,

(b) adequate robustness margins, and

(c) design parameters which can be to adjusted so that performance meets desirable specifications.

In addition, one of the philosophical contributions is the framework handling both state-space based optimization methods and input-output (I/O) analysis methods. Perhaps this will lead the way to more techniques that utilize the best each method has to offer, rather than limiting techniques to only on method.

In chapter 2, basic analysis tools are developed for feedback systems. In section 2.3, basic stability results are presented, including internal stability definitions and results and the incremental stability theorem (showing that incremental stability is equivalent to uniform stability of sets of linearized systems). In section 2.5, closed-loop stability robustness tested are presented, allowing one to determine the robustness of proposed loop operators.

In chapter 3, a very significant result is presented, that of the nondivergence of the extended Kalman filter (EKF) for every detectable plant. This is an important result by itself, as it justifies the use of the EKF for problems where estimation is the end goal itself, without feedback control. In section 3.4.3, guaranteed properties of solutions of certain optimal control problems are presented, generalizing previously known results.

Finally, in chapter 4, the NMBC/LOR methodology is presented in its entirety, with full details.

The development of a unified multivariable nonlinear feedback control design methodology will allow engineers to control systems with much higher performance than current knowledge allows. Global stability guarantees are reassuring, considering the problems encountered by some gain-scheduled designs. Also, certain systems may be so highly nonlinear in all operating regions that linearization based methods offer little hope. Thus it is hoped that the results presented here will be a suitable starting point for a complete methodology useful for practical implementation.

1.1.3 Organization of the Thesis

The thesis is organized into 6 chapters. Chapter 2 covers the analysis of feedback systems. Chapter 3 covers the synthesis of stabilizing control systems. This is done by a separation result (section 3.2) allowing the combination of good estimators (section 3.3) and good state-feedback controllers (section 3.4). Chapter 4 details the entire NMBC/LOR methodology, including all three variations. Chapter 5 presents the results of numerical simulations carried out to test and evaluate the NMBC/LOR techniques on a simple nonlinear pendulum problem. Chapter 6 contains the conclusions and suggestions for future research. The appendices contain two fairly long proofs, details on a promising new observer, the properties of optimal filters, some factorization ideas, and information on state feedback servos.

1.2 Previous Work and Related Literature

In the literature there appear no complete nonlinear control methodologies, in the sense that they are applicable in general to truly nonlinear systems without having all states available for feedback. This section will provide an overview of the background results and control schemes that are available. In addition, an overview of the fairly complete linear theory (LQG/LTR) will be given as a basis for the extensions presented throughout this thesis.

1.2.1 Background theory

The research reported in this thesis draws on several basic feedback theories to provide the machinery and framework for our results.

For stability, input-output (I/O) concepts will be used extensively. The orginal work by Zames [4] and Sandberg [5] dealing with a fairly general nonlinear feedback system provided the groundwork for other I/O stability results (including Safonov [6], Willems [7], Desoer and Vidyasagar [8]). The most general results so far have been Safonov's [6], in which stability and robustness conditions are formulated in terms of sectors and cones in function spaces. The work reported here is closest in spirit to [6].

In addition to the I/O concept of stability, we also will be using Lyapunov concepts (Hahn [9], Krasovskii [10], Vidyasagar [11], and others) to help us tie I/O concepts to the state space. The relationship between Lyapunov and I/O stability has been discussed in Willems [12], Hill and Moylan [13], Vidyasagar and Vannelli [14], and Bodson and Sastry [15].

In the area of performance analysis, the work of Desoer and Wang [16] generalizes some of Bode's [17] original feedback equations to the nonlinear case. The concepts involve relations between loop gain and command following errors, with suitable definitions.

1.2.2 Nonlinear Control Schemes

An ad hoc technique which has been used with some success in aircraft and aircraft engine control is *gain-scheduling*, in which the nonlinear plant is linearized at several operating points and linear controllers designed for each operating point. Then the parameters of the controllers or compensators are scheduled on the basis of some measured variables, as the system changes operating regions. However, there are no a priori guarantees whatsoever as to system stability, robustness, or performance, and some problems have been noted when high performance gain-scheduled designs have been attempted in recent studies (Kapasouris [18], Pfeil [19], and Lively [20]). A solid theoretical treatment of gain-scheduled designs is at present lacking.

Safonov [6] presents conditions under which linear gains in a nonlinear controller will stabilize a nonlinear system, but they are basically bounds on the amount of nonlinearity allowed. In addition, [6] does not address any performance issues, as well as the issue of what to do if the linear gains do not work.

The work on *external linearization* of Meyer and colleagues [21,22,23, et al], by Isidori [24], and Krener, Isidori, and Respondek [25] shows much promise. They seek transformations of state variables and input coordinates that convert a given nonlinear system to a linear one, thus allowing the application of conventional methods. The method is still in its infancy since it requires full state feedback and issues of robustness, performance (command following and disturbance rejection), and dynamic compensation when all state-variables are not available for measurement have not yet been addressed. In addition, not all systems can be linearized in this fashion [22]. A procedure related to external linearization, called *nonlinear decoupling*, is described in [76,77, et al], in which the goal is to reduce (or remove) the coupling of certain pairs of inputs and outputs. There is also a dual procedure to external linearization for designing state estimators (Krener and Respondek [26]) which turns out to be a more difficult proposition. However, if these techniques are able to generate controllers by combining state feedback and observers, the results reported in this thesis will be applicable and useful. We discuss these results in sections 3.3.6 and 3.4.5.

Another proposed nonlinear stabilization scheme is the Q-*parameterization* of Desoer and Lin [27] and Anantharam and Desoer [28], in which the set of controllers stabilizing a given nonlinear plant is suitably parameterized. One drawback to this technique is that the method works only on open-loop stable plants, or on ones that have already been stabilized by some sort of feedback. In addition they do not address the issues of robustness, perfomance, or implementation. In section 4.3.4, we use the ideas of Q-parameterization in our formal loop shaping procedure.

Yet another proposed control scheme is the *describing function* method, in which the plant is converted to a set of linear models, one for each "operating point". Here "operating point" refers to a specific amplitude and frequency of input signal, rather than a point in the state-space. A linear controller is then designed for each "operating point" and the resulting set of controllers is converted to a nonlinear controller through an inverse describing function procedure. This method has much in common with the gain-scheduling procedure and currently has the same drawbacks, namely, no guarantees as to stability, robustness, or performance. Papers on this method using sinusoidal-input describing functions (SIDF) are Taylor [29] and Taylor and Strobel [30]. A similar linearization technique that generates just *one* linear model from the nonlinear system using random-input describing functions (RIDF), called *Nonlinear Quadratic*

Gaussian (NQG) control, is described in Beaman [31]. Describing function ideas may prove useful in the posing and checking of specifications, as we describe in section 2.6.

1.2.3 Linear Multivariable Control Theory

At the start of the research presented in this thesis, LQG/LTR was chosen as the linear multivariable control system design methodology most likely to be extendable to the nonlinear case. It has turned out that virtually all results from the linear theory can be carried over, with only slight changes. This is perhaps a sign of the ease with which future results may be generated.

The LQG/LTR methodology is explained in Doyle and Stein [1], and Stein and Athans [2]. In the LQG/LTR methodology, all robustness and performance specifications are translated into the frequency domain, using singular values [3], which are the multivariable generalization of the classical Bode plots. Once these specifications have been posed, one then designs a "target loop" transfer function which meets these specifications. In LQG/LTR, the target loop is usually designed using optimization theory (the Kalman filter) in order that it have several built-in good properties.

Since the target loop is not a realizable controller by itself, the next step is to modify a special LQG compensator such that the loop transfer function asymptotically approaches the target loop shape. The final compensator is chosen as one that gives an actual loop sufficently close to the target loop over the frequency range of interest, resulting in a final closed-loop system that meets the posed specifications.

In addition to this organized procedure, there are a number of techniques that can be used to help generate a desirable target loop transfer function, allowing a reasonable amount of control over the properties of the final closed-loop system.

1.3 Introduction to the NMBC/LOR Methodology

This section will present a simplified discussion of the methodology proposed in this thesis, termed the Nonlinear Model-Based-Compensator with Loop-Operator-Recovery methodology. There are three variants of the NMBC/LOR technique discussed in chapter 4, but we will discuss them collectively in general terms in this section.

One of the major features of NMBC/LOR is an attempted unification of state-space based optimization methods with input-output analysis. We use state-space techniques for synthesis because gains can be calculated most easily in that framework. However, *unmodeled dynamics* cannot be captured with state-space models, and it is here that I/O methods are most useful; we use I/O methods for analysis.

The steps to design a control system for a particular plant are as follows:

Step 1 (Modeling): Develop a model for the plant as a finite-dimensional nonlinear differential equation. As part of the modeling process develop bounds for the error between the actual plant and the model. Such errors are called *unmodeled dynamics*. The reasons for the discrepancy between the actual plant and the model are (1) unknown dynamics, or (2) known but neglected dynamics.

Step 2 (Specifications): Convert all specifications into specifications on the *loop operator*, which is either the plant cascaded with the compensator, or the compensator cascaded with the plant, depending on the variant of LQR being used. The results in chapter 2 will show us that the loop operator is the important quantity in determining performance and robustness.

Step 3 (Target Loop): Determine a *target loop operator* that meets all of the specifications of step 2. This target loop operator will generally be in a special form, namely a state-feedback loop or a filter loop, as will be discussed in chapter 3. It will be possible to choose this target loop operator in such a way that good robustness and performance properties are guaranteed.

Step 4 (NMBC Construction): Build a Nonlinear Model Based Compensator that guarantees that the closed-loop system will be stable. Chapter 3 discusses the procedure for this.

Step 5 (Loop Operator Recovery): Adjust key parameters in the NMBC design process according to the results of chapter 4, and, with certain technical restrictions, the actual loop operator will approach that of the target loop operator. Use one of these limiting compensators as the final design, so that the loop operator is virtually the same the target loop operator. The final design then meets all of the original specifications.

Remark 1 In the LOR procedure of this last step, the actual loop is forced to look like the target loop. This is done by the compensator creating an *approximate inverse* for the plant and substituting the target loop in its place. Since we desire the final closed-loop system to be closed-loop stable, we must do this in an intelligent manner. For instance, this means making sure that the equivalent of right-half-plane pole-zero cancellations are avoided. Additionally, we see that if the plant were linear, we would have to rule out right-half plane zeros in the plant, as the compensator would not be able to cancel them as other dynamics.

Remark 2 The LOR procedure that we will present in this thesis is practicable: we will present a well-defined algorithm for ensuring the above LOR process. In addition, we will reject certain algorithms as being computationally too difficult (e.g. the optimal filter in section 3.3.2) and thus arrive at concrete algorithms that are within the reach of current technology.

This completes the outline of the NMBC/LOR procedure. The rest of the thesis is devoted to developing all of these ideas in detail, with mathematically precise results.

1.4 Notation

$:=$	"is defined as"
I	The identity matrix or operator
0	The zero matrix or operator
$\mathbb{R}$	The real numbers
$\mathbb{R}^n$	space of ordered n-tuples of real numbers
$\mathbb{R}_+$	The non-negative real numbers
∇g	The gradient matrix of the function $g:\mathbb{R}^n \to \mathbb{R}^m$
$\lvert x\rvert$	The Euclidean norm of the vector x, e.g $(x^Tx)^{1/2}$
$\lvert A\rvert, \sigma_{max}[A]$	The maximum singular value of the matrix A
$\sigma_{min}[A]$	The minimum singular value of the matrix A
$\mathcal{L}_p$	signal space with elements of finite p-norm
$\mathcal{L}$	extended signal space $= \{x:\mathbb{R}_+ \to :\mathbb{R}^n \mid \lVert x\rVert_{\infty,\tau} < \infty \ \forall \tau \in \mathbb{R}_+\}$
P_τ	truncation operator
$\lVert x\rVert_p$	p-norm of signal $x(\cdot)$ as a member of $\mathcal{L}$
$\lVert x\rVert_{p,\tau}$	truncated p-norm of signal $x(\cdot)$, $= \lVert P_\tau x\rVert_p$
$\lVert (x,y)\rVert$	see section 2.2
Φ	plant dynamics operator $= [S^{-1} - F]^{-1}$
$\Phi(t,\tau)$	state transition matrix for a linear time-varying system
$A>B$ $(A\geq B)$	the matrix A-B is positive (semi)-definite
A^T, x^T	the transpose of the matrix A or vector x
P	the plant operator
K	the compensator operator
T	the loop operator

CHAPTER 2. ANALYSIS: STABILITY, PERFORMANCE & ROBUSTNESS

2.1 Introduction

In order to design good feedback systems we must first understand what properties a "good" feedback system should have. This chapter presents this information on the basic definitions and techniques that we will be using for the analysis of multivariable nonlinear feedback loops. We start in section 2.2 with the definition of our plant model and basic concepts. Section 2.3 then concentrates on the issue of stability, including closed-loop equations, incremental stability, and the relationship between input-output (I/O) stability and Lyapunov (or zero-input) stability. The analysis development is continued in section 2.4, where the command following and disturbance rejection performance of a feedback system is analyzed, and in section 2.5, where tests for stability robustness with respect to unstructured unmodeled dynamics are presented. Finally, the chapter concludes in section 2.6 with a discussion of the issues involved in actually carrying out some of these tests.

We will try to parallel the linear system theory [1,2,3] development as much as possible to give the reader more of a feeling for the analogy between results for linear and nonlinear systems. In addition, we will endeavor to use the same notation wherever possible.

2.2 Models and Basic Definitions

In this section we define the model of the plant to be controlled and our definitions for norms, stability, etc. We will also present the operator notation for the description of dynamical systems and associated notation.

We first define the so called *design plant model* that we shall use. The design plant model will be the model used to apply the NMBC/LOR methodology in chapter 4 and includes all scaling, normalizations, and augmented dynamics, such as integrators, as discussed in section 4.4. Our design plant model will be described by

$$\dot{x}(t) = f(x(t)) + B\,u(t); \quad x(0) = 0 \tag{2.1a}$$

$$y(t) = C\,x(t) \tag{2.1b}$$

where $x(t)\in\mathbb{R}^n$ is the *state*, $u(t)\in\mathbb{R}^m$ is the *input*, and $y(t)\in\mathbb{R}^m$ is the *output*. B is an n×m matrix and C is an m×n matrix. We assume that the nonlinearity $f:\mathbb{R}^n\to\mathbb{R}^n$ is at least twice continuously differentiable, with $f(0)=0$, and we will usually assume that there exists M_f such that

$$|\nabla f(x)| \le M_f \qquad \text{for all } x\in\mathbb{R}^n \tag{2.2a}$$

$$\left|\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k}\right| \le M_f \text{ for all } x\in\mathbb{R}^n,\ 0\le i,j,k \le n. \tag{2.2b}$$

In (2.1) the initial condition for the state is zero. In general this is how we will deal with differential equations from an input-output viewpoint. If the system is controllable, then clearly we can access all possible behavior of (2.1) by first traveling to a desired state, then starting our observation. When we use Lyapunov techniques, we will use a nonzero initial condition for the plant model; section 2.3 contains results relating the two formulations.

The model (2.1) is more general than it might appear, for suppose we had a system

$$\dot{z} = g(z,v) \tag{2.3a}$$

$$y = h(z) \tag{2.3b}$$

with v the input to the system. Through a change of state variables, it will usually be possible to make the output map h linear. The simplest way to do this is to make the first part of the transformed state vector, x, be y and the rest be whatever states are needed to make up a complete state vector. Then the new system (with the same input-output characteristics) will have a linear output map. Now, if we add integrators at the input to (2.3), i.e. define a new control input u so that $\dot{v} = u$, we will have transformed the system (2.3) into the form (2.1). Since augmenting a system with integrators is almost always done to improve low frequency performance and to ensure zero steady-state errors to step commands and/or disturbances, the model (2.1) can actually be used to handle a very wide class of original models (2.3). Some of the results presented in this thesis do not require that the input and output maps be linear, but their linearity will be assumed here to simplify equations as well as allow all the results to apply to a specific model (2.1). It will usually be clear from the proof of a result whether linearity is required for that specific theorem.

In addition to the description (2.1) for the model, we must include a description of model uncertainty. We will represent model uncertainty by *unstructured unmodeled dynamics*, for which we will assume that we have an I/O bound of some type. Section 2.5 has details on this topic.

We now consider the I/O viewpoint for systems, in which a system is thought of as rule for mapping inputs into outputs. Here inputs and outputs are entire signals, i.e. trajectories, not just elements of $\mathbb{R}^n$. We call a set of signals a *signal space*, and a rule for mapping one signal space into another is an *operator*. Since we want to be able to make quantitative statements, we need a way of assigning sizes to these signals (elements of a signal space). One way to do this is by the use of norms.

Definition For $1 \le p < \infty$, we define the p-norm of a signal $x:\mathbb{R}_+ \to \mathbb{R}^n$

$$\|x\|_p = \left[\int_0^\infty |x(t)|^p \, dt \right]^{1/p}. \tag{2.4}$$

For $p=\infty$ we use

$$\|x\|_\infty = \sup_t |x(t)|. \tag{2.5}$$

These definitions of course are not finite for all functions $x:\mathbb{R}_+\to\mathbb{R}^n$. We will restrict the signals on which we apply these norms as follows.

Definition $\mathcal{L}_p^n$ is the set of all signals $x:\mathbb{R}_+\to\mathbb{R}^n$ for which $\|x(t)\|_p$ is finite, i.e.

$$\mathcal{L}_p^n = \left\{ x:\mathbb{R}_+\to\mathbb{R}^n \;\middle|\; \|x\|_p < +\infty \right\} \tag{2.6}$$

In functional analysis, values of p are usually considered for the full range $[1,\infty]$. In this thesis we will be concerned primarily with the cases for $p=2$ and $p=\infty$. Since we restricted the set $\mathcal{L}_p^n$, it is not quite large enough to deal with all of our system theory questions because it does not include any signals that "blow up", or grow without bound. Without these types of signals, we cannot discuss unstable systems, and thus stability itself remains inaccessible. To be able to handle these growing signals, we must *extend* the set $\mathcal{L}_2^n$ by the following mechanism. For more details see [4,6,7,8].

Definition The *truncation operator* P_τ is defined by its operation on an arbitrary signal $x:\mathbb{R}_+\to\mathbb{R}^n$ as

$$(P_\tau x)(t) := \begin{cases} x(t) & \text{if } t \leq \tau \\ 0 & \text{if } t > \tau \end{cases}. \tag{2.7}$$

Definition The *extended space* $\mathcal{L}_{p,e}^n$ is the set of signals whose truncations lie in $\mathcal{L}_p^n$, i.e.

$$\mathcal{L}_{p,e}^n := \{ x:\mathbb{R}_+\to\mathbb{R}^n \mid P_\tau x \in \mathcal{L}_p^n \;\; \forall\, \tau > 0 \}. \tag{2.8}$$

We will frequently drop the superscript n, as the dimension of the underlying vector space is usually quite apparent. In addition, we will want exclude some signals with very bizarre nonphysical behavior. For example, consider

$$x(t)=\begin{cases} t^{-1/4} & t < 1 \\ t^{-2} & t \geq 1 \end{cases} \tag{2.9}$$

which goes to infinity at t=0 and in addition (believe it or not) belongs to $\mathcal{L}_2$. We eliminate this type of annoyance by only considering the set $\mathcal{L}_{\infty,e}$ for the rest of this thesis. For simplicity, we will define the set $\mathcal{L} := \mathcal{L}_{\infty,e}$. As a note to the mathematically sophisticated reader, we will not be concerning ourselves with the behavior of signals on sets of zero measure, as this does not affect smooth physical systems.

Remark The above mathematics is just one possible way to utilize the concepts of extended spaces and so on. In fact, extensions to discrete time systems are quite easy [4,6]. We restrict ourselves here in order to give a more concrete flavor, reduce technical restrictions, and to tie results to the state-space domain.

The operator description of a nonlinear system is simply a mapping $P:\mathcal{L}\rightarrow\mathcal{L}$. For example, we write

$$y = Pu; \quad u,y\in\mathcal{L} \tag{2.10}$$

to mean that the input u produces the output y. Remember that u and y are not points in $\mathbb{R}^n$ but are entire trajectories in $\mathbb{R}^n$, i.e.

elements of $\mathcal{L}$. The value of the response of the system P to the input u at time t is given by

$$y(t)=(Pu)(t). \tag{2.11}$$

We will assume that P0=0 for all operators we will be considering. This does not cause any loss in generality, as the 0 input response can be dealt with separately. We define the addition and composition of operators in the expected way:

$$(A+B)u \;:=\; Au + Bu \tag{2.12}$$

$$ABu \;:=\; A(Bu) \tag{2.13}$$

We are now able to extend the notion of size to signals in $\mathcal{L}$ and to operators:

Definition The *truncated* L_p*-norms* of $x\in\mathcal{L}$ are

$$\|x\|_{p,\tau} \;:=\; \|P_\tau x\|_p \;=\; \left[\int_0^\tau |x(t)|^p\,dt\right]^{1/p} \;;\quad p<\infty \tag{2.14a}$$

$$\|x\|_{\infty,\tau} \;:=\; \|P_\tau x\|_\infty \;=\; \sup_{0\le t\le\tau} |x(t)| \tag{2.14b}$$

Definition The L_p*-norm*, or *gain*, of an operator (system) is

$$\|P\|_p \;:=\; \sup \frac{\|Pu\|_{p,\tau}}{\|u\|_{p,\tau}} \tag{2.15}$$

where the supremum is taken over all $u\in\mathcal{L}$ and all $\tau>0$. If the type (i.e. p) is not specified, then results hold for all p-norms, consistently throughout a discussion. In words, the gain is thelargest possible amplification in signal size that can be achieved over all possible inputs. Similarly, we have

Definition The L_p-*incremental gain* of an operator is

$$\|P\|_{p,\Delta} := \sup \frac{\|Pu_1 - Pu_2\|_{p,\tau}}{\|u_1 - u_2\|_{p,\tau}} \tag{2.16}$$

where the supremum is taken over all $u_1, u_2 \in \mathcal{L}$ and all $\tau > 0$.

Definition An operator (system) P is L_p-*stable* if it has finite gain, i.e. $\|P\|_p < +\infty$.

Definition An operator P is L_p-*incrementally stable* if it has finite incremental gain, i.e. $\|P\|_{p,\Delta} < +\infty$.

Note that a system P is stable if and only if there exists a constant k such that

$$\|Pu\|_\tau \leq k\ \|u\|_\tau \qquad ;\ \forall u \in \mathcal{L}, \tau \in \mathbb{R}_+ \tag{2.17}$$

and that the smallest such k is the gain $\|P\|$ of the system.

Remark We define stability here because there is no standard definition. Other possibilities include using some increasing function instead of a linear gain k in (2.17), and not requiring the output to have zero norm when the input is zero. Note that in the time-invariant linear case the types of stability above are all equivalent to the standard one.

As we will occasionally have need to discuss the size of the vector $z=(x,y)$, with $z\in\mathbb{R}^{n+m}$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, or the signal $z=(x,y)$ with $z\in\mathcal{L}^{n+m}$, $x\in\mathcal{L}^n$, $y\in\mathcal{L}^m$, we clarify the issue by defining:

$$|(x,y)| \;:=\; [\;|x|^2 + |y|^2\;]^{1/2} \tag{2.18}$$

$$\|(x,y)\|_{2,\tau} \;:=\; [\;\|x\|_{2,\tau}^2 \;+\; \|y\|_{2,\tau}^2\;]^{1/2} \tag{2.19a}$$

$$\|(x,y)\|_{\infty,\tau} \;:=\; \|x\|_{\infty,\tau} \;+\; \|y\|_{\infty,\tau}. \tag{2.19b}$$

Technically, this last definition is not consistent with the definition of a signal given previously, in the sense that if $z=(x,y)$, we have

$$\|z\|_{2,\tau} = \|(x,y)\|_{2,\tau} \tag{2.20}$$

but only

$$\|z\|_{\infty,\tau} = \sup_{t\leq\tau} |z(t)| \leq \|(x,y)\|_{\infty,\tau} \tag{2.21}$$

with equality not guaranteed in general. To fix this we would have to redefine the norm of a vector in $\mathbb{R}^n$ just for the L_∞ case. However, this is not worth it because the definition given above is sufficient for our purposes, since

$$\|z\|_{\infty,\tau} \;\leq\; \|(x,y)\|_{\infty,\tau} \;\leq\; 2\;\|z\|_{\infty,\tau} \tag{2.22}$$

and we are generally just concerned with the existence of bounds, not their exact value.

We make one more shorthand notational definition:

Definition The closed-ball B_h is defined as the set

$$B_h \ := \ \{ \ x \in \mathbb{R}^n \ \Big| \ |x| \leq h \ \}. \tag{2.23}$$

To simplify equations, we will now define a special nonlinear operator Φ by the mapping from w to x given by

$$\dot{x}(t) = f(x(t)) + w; \quad x(0) = 0 \tag{2.24}$$

and shown in the block diagram of figure 2-1. If we let F be the nondynamical operator defined by

$$(Fx)(t) := f(x(t)) \tag{2.25}$$

and S be the integral operator, we can write

$$\Phi := [S^{-1} - F]^{-1} \tag{2.26}$$

We can now see the usefulness of Φ; our plant (2.1) can now be written in compact form

$$y = Pu; \qquad P = C\Phi B \tag{2.27}$$

This operator representation of our plant will be very useful throughout the rest of the thesis. Note that for (2.27) to hold, neither B nor C need be linear.

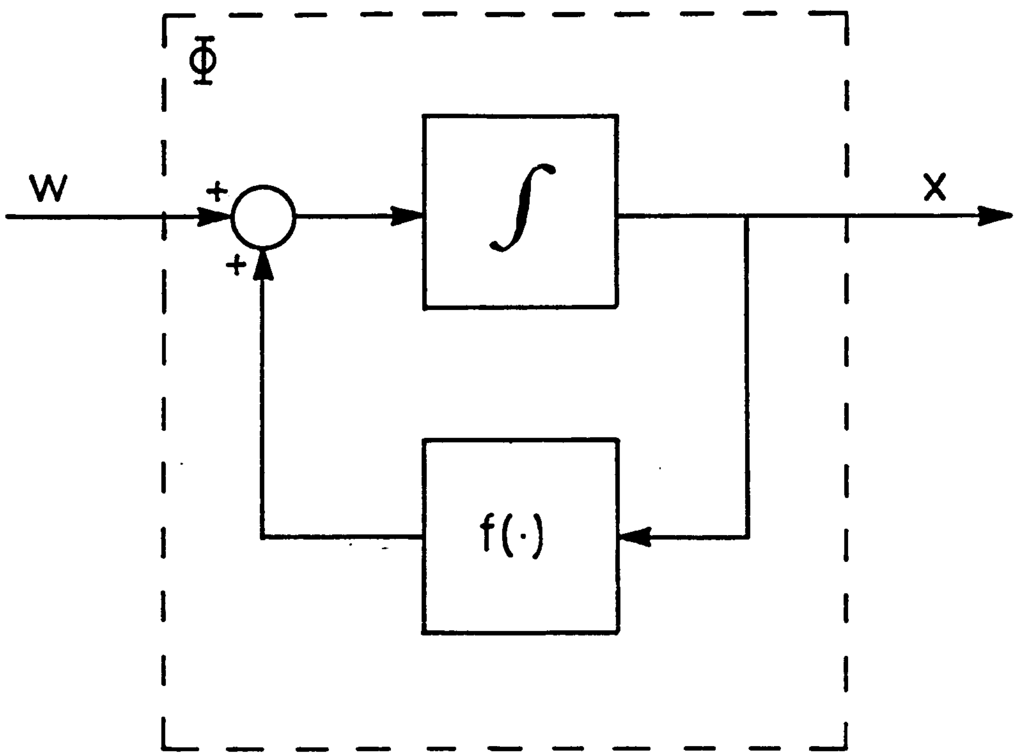


Figure 2-1: The Φ Operator

All of the preliminary mathematics has now been covered and we are ready to go on with the analysis of nonlinear systems. In particular the next section will discuss stability in more detail.

2.3 Stability and Feedback

In this section we investigate the issue of stability more closely. We start in section 2.3.1 with a derivation of the closed loop equations for a feedback system, using operator notation. In section 2.3.2 we then examine the issue of internal stability, a very important issue for the implementation of a control system. In section 2.3.3 the relation between certain types of Lyapunov stability and I/O stability is detailed. Finally, in section 2.3.4 incremental stability is examined in detail. Incremental stability will turn out to be an important ingredient in the synthesis of state estimators.

2.3.1 The Closed Loop Equations

We now examine the use of operators for analyzing closed loop feedback systems. Consider the closed loop system of figure 2-2, where we have a plant P and compensator K with a command input r, input disturbance w, and a output disturbance d. This arrangement is certainly not the most general possible; we could for instance have the compensator inputs include the measured signal y and the reference signal r directly, instead of just their difference. We choose this framework because it happens to fit the results that come later, as well as the fact that it is a good starting point for

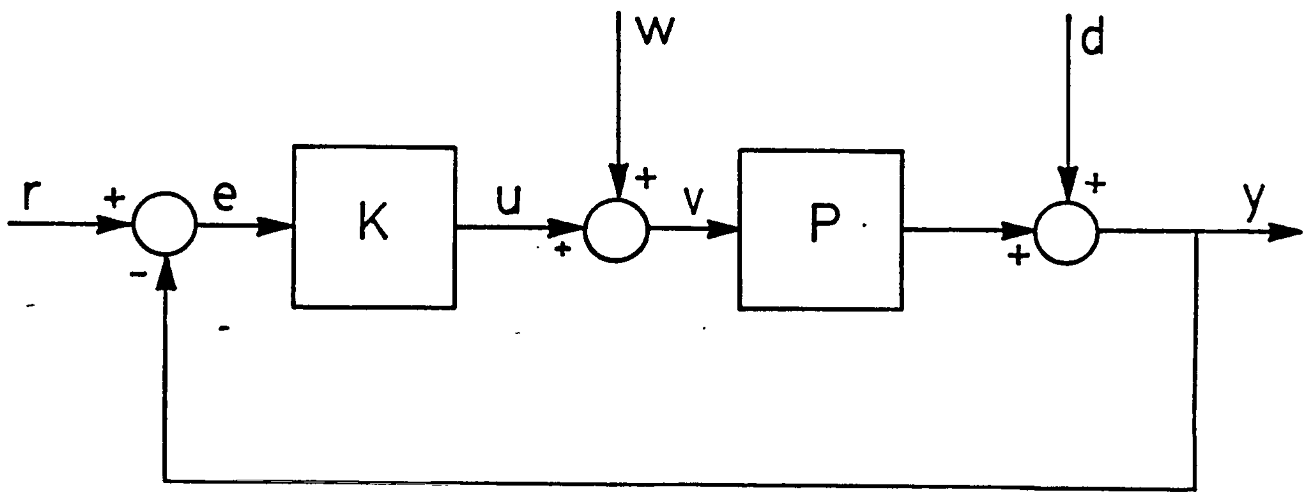


Figure 2-2: Closed-Loop System

discussion. In addition, the results of [32] indicate that such a structure is reasonable in the sense that if the system shown in figure 2-2 can be stabilized, then a "stable factorization" of the plant exists. We show in the appendix how the existence of a stabilizing state feedback function (see sections 3.2 and 3.4) is related to the idea of factorization.

Continuing to analyze the configuration of figure 2-2 we make the following definition:

Definition The operator PK is called the *loop operator broken at the plant output*. (-K)(-P) is the *loop operator broken at the plant input*. The word "broken" is optional.

The terminology should be clear by analogy with the linear systems case. We will frequently use the terms *loop operator* or *loop*, when it is clear from the context which loop we mean. Letting T=PK be the loop broken at the plant output, the loop equations for figure 2-2 when w=0 are

$$e = r - d - Te \tag{2.28a}$$

$$y = d + Te \tag{2.28b}$$

Assuming I+T invertible, we can write

$$e = [I+T]^{-1}(r-d) \tag{2.29}$$

$$y = d + T[I+T]^{-1}(r-d). \tag{2.30}$$

The question of invertibility of I+T is relatively easy to answer here. We basically require T to have no "instantaneous gain"; our plant model (2.1) guarantees this. [7] has a very detailed exposition of this concept.

We can thus see that the stability of the closed loop system when w=0 depends only on the stability of $[I+T]^{-1}$, since

$$T[I+T]^{-1} = I - [I+T]^{-1}. \tag{2.31}$$

The situation when w is nonzero is somewhat more difficult. For instance, we have assumed nothing in the above discussion that would rule out the analog of pole/zero cancellations in the right-half-plane. Thus we really have not guaranteed any sort of practical closed-loop stability by making $[I+T]^{-1}$ stable. The type of thing we need is an extension of the closed-loop matrix used [33] in the linear situation in which both $[I+PK]^{-1}$ and $[I+KP]^{-1}$ are required to be stable. We can do that by the following definition, modified for command following from [6].

Definition The system in figure 2-2 will be said to be *closed-loop stable* if the mapping $(w,d,r) \mapsto (y,u)$ is stable. Because d and r are added together in this configuration, this is easily shown to be equivalent to the stability of either of the mappings (with d=0)

$$(w,r) \mapsto (e,v) \tag{2.32}$$

$$(w,r) \mapsto (y,u). \tag{2.33}$$

In sections 2.4 and 2.5 we will examine the closed loop system performance and robustness in terms of the loop operators.

2.3.2 Internal Stability

Suppose that we have a system P that is stable but we are concerned about *internal stability*, that is, whether or not the insides, or states, of the system are stable. More precisely:

Definition A system P described by (2.1) is *internally stable* if the mapping $u \mapsto x$ is stable, i.e. there exists a constant k such that

$$\|x\|_{\tau} \leq k \; \|u\|_{\tau} \qquad \forall \; u \in \mathcal{L}, \; \tau \in \mathbb{R}_{+}. \tag{2.34}$$

For linear systems, one can have I/O stability without internal stability only if the system has pole-zero cancellations in the right half plane and thus is unobservable and uncontrollable. This situation can be avoided, as any realistic model that we want to control should be both observable and controllable, or at least stabilizable and detectable (the minimum required for closed-loop stability). In analogy with the linear case, we therefore define one possible way to rule out such behavior.

Just for the remainder of this section, we will associate a particular realization with a system P. That is, think of P as a label for the set of equations (2.1). This slight abuse of notation will save us the trouble of saying "a particular realization of P, given by equation (2.1)" instead of just "P".

Definition We shall say $P \in \mathcal{SD}$ (for Set of Detectable plants) if there exist constants k_1 and k_2 such that

$$\|x\|_\tau \le k_1\|y\|_\tau + k_2\|u\|_\tau \qquad \forall\ u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.35}$$

The reason that this is a useful definition is that the $\mathcal{SD}$ property is retained under feedback and guarantees that we will have internal stability if we have stability. We will show in section 3.2 that if estimators can be built for P, then $P\in\mathcal{SD}$. The set SD is related to the concept of observability of other researchers [12,78]. The following are some easy results:

Theorem 2.1

(a) If $P\in\mathcal{SD}$ then $P[I+P]^{-1}\in\mathcal{SD}$, where $P[I+P]^{-1}$ is realized in the expected way, i.e. $u=u_1-y$, where u_1 is the new input.

(b) If $P\in\mathcal{SD}$ and P is stable then P is internally stable.

(c) If P is internally stable then $P\in\mathcal{SD}$.

Proof

(a) Let $P\in\mathcal{SD}$. The closed loop configuration gives $y = P(u_1-y)$ so that

$$\begin{aligned}\|x\|_\tau &\le k_1\|y\|_\tau + k_2\|u_1-y\|_\tau \\ &\le (k_1+k_2)\|y\|_\tau + k_2\|u_1\|_\tau; \quad \forall\tau\in\mathbb{R}_+ \end{aligned}\tag{2.36}$$

and therefore the closed loop system $P[I+P]^{-1}\in\mathcal{SD}$.

(b) Let P be stable and $P \in \mathcal{SD}$. Then

$$\|x\|_T \le k_1\|y\|_T + k_2\|u\|_T \ ; \ \forall T \in \mathbb{R}_+$$
$$\le (k_1\|P\| + k_2)\|u\|_T ; \ \forall T \in \mathbb{R}_+ \qquad (2.37)$$

and therefore P is internally stable.

(c) Simply pick $k_1=0$. ∎

From this list of properties, it appears that the set $\mathcal{SD}$ is quite useful. Membership in the set guarantees that internal stability is gotten for free with stability. Property (a) guarantees that if we design a compensator to stabilize a system so that the open loop dynamics are in $\mathcal{SD}$, the closed loop will also be in $\mathcal{SD}$ and thus ensure internal stability.

2.3.3 Lyapunov and I/O Stability

In this section we discuss the relationship between Lyapunov and I/O stability. The basic references for this section are [12,13,14,15]. We will present the results in their most general form so they will be most applicable throughout the rest of the thesis. In general, we will be concerned about the relationship between the I/O behavior of the system

$$\dot{x} = f(x,u,t); \quad x(0)=0 \qquad (2.38)$$

and the zero input behavior of

$$\dot{x} = f(x,0,t); \ x(0)=x_0. \qquad (2.39)$$

We assume that f is locally Lipshitz, i.e. there exist closed balls $B_h, B_\epsilon \subset \mathbb{R}^n$, and constants M_f, M_u such that for all $x_1, x_2 \in B_h$, $u_1, u_2 \in B_\epsilon$

$$|f(x_1,u_1,t)-f(x_2,u_2,t)| \leq M_f \; |x_1-x_2| + M_u \; |u_1-u_2|; \quad \forall t \geq 0. \tag{2.40}$$

This guarantees the local existence and uniqueness of solutions to (2.38) and (2.39). The following is a theorem from [15].

Theorem 2.2 Let f be locally Lipshitz as above (2.40) and assume that x=0 is a stable equilibrium point of (2.39), i.e. f(0,0,t)=0 for all $t \geq 0$, and there exists a ball $B_\delta \subset B_h$ such that, for all $x_0 \in B_\delta$, $t_0 \geq 0$, and $t \geq t_0$, $x(t) \in B_h$ along solutions of (2.39) starting at x_0. Then the following statements are equivalent:

(a) x=0 is an *exponentially stable* equilibrium point of the system (2.39), i.e. there exist α, M, such that for all $x_0 \in B_h \subset \mathbb{R}^n$, $t_0 \geq 0$,

$$|x(t)| \leq M \; |x_0| \; e^{-\alpha(t-t_0)} \;\; ; \quad \forall t \geq t_0. \tag{2.41}$$

(b) there exists a function v(x,t), and constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all $x \in B_\delta$, $t \geq 0$,

$$\alpha_1^2 |x|^2 \leq v(x,t) \leq \alpha_2^2 |x|^2 \tag{2.42}$$

$$\frac{dv(x,t)}{dt} \leq -\alpha_3 |x|^2 \tag{2.43}$$

$$\left|\frac{\partial v(x,t)}{\partial x}\right| \leq \alpha_4 |x| \; . \tag{2.44}$$

Furthermore, (a) implies the requirement of stability of the origin and thus implies (b) without that a priori requirement.

Remark This theorem is known as one of the converse thereoms of Lyapunov function theory [9,10]. The function v is a Lyapunov function for the system (2.39); from Lyapunov theory, if such a function exists the system is Lyapunov stable, i.e. the state tends to zero. The theorem states that if v obeys some asymptotic properties (2.42-2.44), the stability is exponential. Note that not every Lyapunov function for (2.39) will obey (2.42-2.44), even if the system is exponentially stable; however, there will exist at least *one* with such properties.

We are now ready to state a theorem which relates exponential Lyapunov stability to I/O stability, modified slightly from [14,15].

Theorem 2.3 (Small-Signal Stability) Suppose that x=0 is an exponentially stable equilibrium point of (2.39) and that the Lipshitz condition (2.40) holds. Then system (2.38) is small-signal L_p-stable, i.e. for all $p \in [1,\infty]$, there exist constants γ_p and c_∞ such that whenever $x(0)=0$, and $|u(t)| \leq c_\infty \ \forall t$, we have that

$$\|x\|_{p,\tau} \leq \gamma_p \|u\|_{p,\tau}. \tag{2.45}$$

Furthermore, the constants are given by

$$\gamma_\infty = M_u \cdot \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3} \tag{2.46}$$

$$\gamma_p = \gamma_\infty \cdot \frac{\alpha_2}{\alpha_1}\ ; \qquad p < \infty \tag{2.47}$$

$$c_\infty = \min\{\epsilon, \frac{1}{\gamma_\infty}\min\{h,\delta\}\}, \tag{2.48}$$

where properties (2.42-2.44) hold in B_δ.

This result shows that if we can prove some things in the state space framework (Lyapunov stability) we can translate those results to the I/O domain. There are results [12] that discuss the converse of this theorem, which we will not present rigorously as they will not be needed later. Roughly, they say that a reachable (controllable) and observable realization of an I/O stable system is globally asymptotically stable and bounded. Thus, it appears that the concepts of zero-input Lyapunov, or state stability, are quite closely related to the concept of I/O stability, once we remove anomalies like uncontrollable and unobservable states. This is intuitively pleasing, since the two notions come from widely different viewpoints.

2.3.4 Incremental Stability

Incremental stability is related to the concept of continuity on signal spaces. The output of an incrementally stable system changes by an amount not more than proportional to the change in the input signal. In equations, it says that there exists a k such that for any two inputs u_1 and u_2, the respective outputs y_1 and y_2 must obey

$$\|y_1-y_2\|_T \leq k \|u_1-u_2\|_T \qquad \forall\, T\in\mathbb{R}_+ \tag{2.49}$$

Consider the situation where u_1 and u_2 are very close to each other. This should remind us of small-signal linearization and we would expect that the concepts are related. They are.

Theorem 2.4 Let ∇f obey the restriction (2.2). Let the nonlinear system $x=Pu$ be described by

$$\dot{x}(t) = f(x(t),t) + u(t). \tag{2.50}$$

Then P is L_∞ incrementally stable i.e.

$$\|Pu_1-Pu_2\|_{\infty,T} \leq k \|u_1-u_2\|_{\infty,T} \quad \forall u_1,u_2\in\mathcal{L},T\in\mathbb{R}_+ \tag{2.51}$$

if and only if the linearized system ∇P_x described by

$$\dot{\xi}(t) = \nabla f(x(t),t)\xi(t) + w(t) \tag{2.52}$$

is uniformly L_∞-stable for all possible trajectories $x\in\mathcal{L}$, i.e.

$$\|\xi\|_{\infty,T} \leq k \|w\|_{\infty,T}; \quad \forall\, w\in\mathcal{L},T\in\mathbb{R}_+ \tag{2.53}$$

for all possible trajectories $x\in\mathcal{L}$.

Proof See appendix A.

Remark This theorem requires that the B matrix in both the linearized and nonlinear equations be the identity. We could do away with this restriction if we used some notion of controllability, so that all the behavior of the system could be accessed through the B matrix. This is related to the exponential stability of the previous section, in which stability with an identity B matrix was guaranteed.

This theorem will be very useful in the next chapter where we will show that filtering is tied closely to the notion of incremental stability. There we will extend this theorem somewhat to cover the L_2 case, because of the properties of the extended Kalman filter.

2.4 Closed-Loop Performance

This section will analyze the performance of the closed-loop system of figure 2-2. The loop operator T=PK was briefly discussed in section 2.3.1. Here we judge the performance by how well the system follows commands and rejects disturbances, i.e. by how small e=r-y is. For this case, we set w=0. Let

$$H_{yr} := T[I+T]^{-1} \tag{2.54}$$

be the map from r to y in the closed-loop system, when d=0. Then, as derived in section 2.3.1

$$y = d + H_{yr}(r-d) \tag{2.55}$$

Theorem 2.5 [16] If, for all $r \in \mathfrak{R} \subset \mathcal{L}$ and for all $d \in \mathfrak{D} \subset \mathcal{L}$

$$\|[I+T]^{-1}(r-d)\|_\tau \ll \|r-d\|_\tau; \quad \forall \tau \in \mathbb{R}_+, \tag{2.56}$$

then $e = r-y \cong 0$ on $\mathfrak{R}$ and $\mathfrak{D}$ in the sense that

$$\|e\|_\tau = \|r-y\|_\tau \ll \|r\|_\tau + \|d\|_\tau; \ \forall r \in \mathfrak{R}, d \in \mathfrak{D}, \tau \in \mathbb{R}_+. \tag{2.57}$$

Proof See [16]

Remark 1 This theorem shows the linearizing effect of high gain feedback (an identity map is linear). It is a generalization of the original work of Bode [17] for linear systems, and can be stated in a much more general way, where instead of unity feedback as in figure 2-2, there is an operator F [16]. In that case, the conclusion for high gains is, as expected, that $H_{yr} \cong F^{-1}$.

Remark 2 The sets $\mathfrak{R}$ and $\mathfrak{D}$ are analogous to frequency ranges in the linear theory situation, but present more of a problem. The test (2.56) must be checked for *every* signal in the sets $\mathfrak{R}$ and $\mathfrak{D}$; we are not so lucky as to have all the signals of a given frequency but varying amplitudes give the same behavior. The basic idea in using theorem 2.5 is to think of $\mathfrak{R}$ and $\mathfrak{D}$ containing the signals for which good command following and disturbance rejection are desired. Typically, these will be all the "low frequency" signals. Section 2.6 will discuss these issues in more detail.

Remark 3 Theorem 2.5 also shows that we selected a useful definition for stability, because we have the result that our stability with high gain produces small errors, which is a fundamental requirement.

We can use theorem 2.5 to relate the error magnitude to the open loop gain, as in the self-explanatory result:

Theorem 2.6 Let $u = [I+T]^{-1}x$. Then

$$\frac{\|[I+T]^{-1}x\|_T}{\|x\|_T} = \frac{\|u\|_T}{\|[I+T]u\|_T} \leq \frac{\|u\|_T}{\|Tu\|_T - \|u\|_T}$$

$$= \left[\frac{\|Tu\|_T}{\|u\|_T} - 1 \right]^{-1} . \tag{2.58}$$

Thus we see that we can relate the size of the sensitivity (or error) operator, $[I+T]^{-1}$, to the size of the open-loop gain. The engineering rule-of- thumb which tells us that we need a loop gain of, say, greater than 100 for errors of less than 1% still applies.

2.5 Closed-Loop Stability Robustness

This section will investigate the issue of stability robustness. Roughly speaking, stability robustness refers to the amount of perturbation that a nominally stable closed-loop system can withstand and still be guaranteed stable. We want to be able to check that our closed-loop designs are robustly stable, and if they are not, would like some help in redesigning them so that they are in fact robust with respect to closed-loop stability.

In the following theorems, we will be assuming that we have a nominal loop operator, T, that we know to be closed-loop stable, along with a particular characterization of the actual loop T_A in terms of the modeling error, E. Obviously, we never really know what the actual loop T_A is, and that is the whole point of the robustness tests: If we can bound the error somehow, then we may be able to guarantee the closed-loop stability of the actual system, *without knowing* T_A *explicitly*.

Theorem 2.6 (Robustness Tests)

(a) (Additive Error) Suppose that T is closed-loop stable, i.e. $[I+T]^{-1}$ is stable, and

$$T_A = T + E. \tag{2.59}$$

Then T_A will be closed-loop stable if there exists a $0<\delta<1$ such that

$$\|Eu\|_\tau \leq \delta \ \|(I+T)u\|_\tau; \quad \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.60}$$

(b) (Division Error) Suppose that T is closed loop stable and

$$T_A = T(I+E)^{-1}. \tag{2.61}$$

Then T_A will be closed-loop stable if there exists a $0<\delta<1$ such that

$$\|Eu\|_\tau \leq \delta \ \|(I+T)u\|_\tau; \quad \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+ \tag{2.62}$$

and

$$\|E\| < \infty. \tag{2.63}$$

(c) (Subtractive Error) Let T be closed-loop stable,

$$E = T_A^{-1} - T^{-1}, \tag{2.64}$$

let ET be stable, and let the set $\mathcal{G}$ be defined by

$$\mathcal{G} := \{x = Tu \mid u\in\mathcal{L} \}. \tag{2.65}$$

Then T_A will be closed-loop stable if there exists $0<\delta<1$ such that

$$\|ETu\|_\tau \le \delta \; \|(I+T)u\|_\tau; \quad \forall u \in \mathcal{L}, \tau \in \mathbb{R}_+, \tag{2.66}$$

or

$$\|Eu\|_\tau \le \delta \; \|(I+T^{-1})u\|_\tau; \quad \forall u \in \mathcal{G}, \tau \in \mathbb{R}_+. \tag{2.67}$$

(d) (Multiplicative Error) Let T be closed-loop stable,

$$T_A = (I+E)T, \tag{2.68}$$

and the set $\mathcal{G}$ be defined by

$$\mathcal{G} := \{x = Tu \mid u \in \mathcal{L}\} \tag{2.69}$$

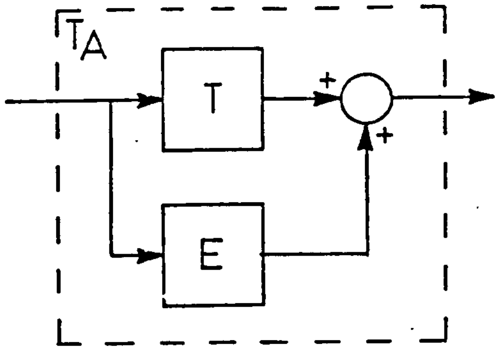
Then T_A will be closed-loop stable if there exists $0<\delta<1$ such that either

$$\|ETu\|_\tau \le \|(I+T)u\|_\tau; \quad \forall u \in \mathcal{L}, \tau \in \mathbb{R}_+, \tag{2.70}$$

or

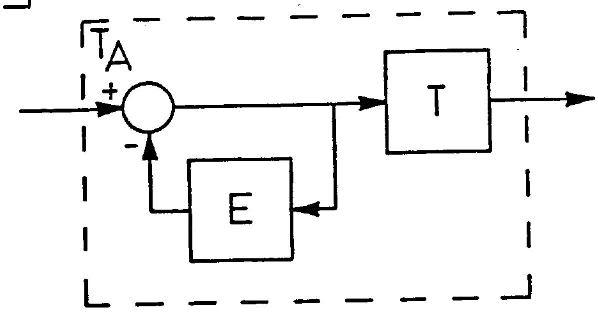
$$\|Eu\|_\tau \le \|(I+T^{-1})u\|_\tau; \quad \forall u \in \mathcal{G}, \tau \in \mathbb{R}_+. \tag{2.71}$$

Figure 2-3 gives an interpretation of the error E for each of these representations (a-d). In each case, the dashed box represents the actual plant T_A for each definition of the error E.



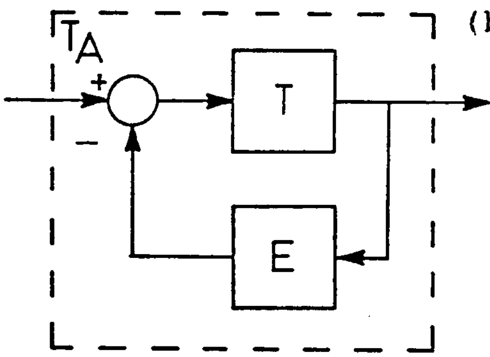
$T_A = T + E$

(a) Additive Error



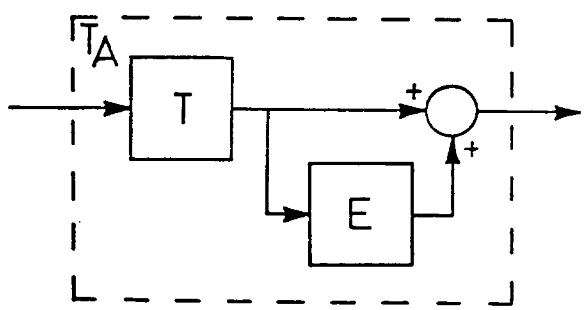
$T_A = T[I+E]^{-1}$

(b) Division Error



$T_A = [E+T^{-1}]^{-1}$

(c) Subtractive Error



$T_A = [I+E]T$

(d) Multiplicative Error

Figure 2-3: Robustness Tests

Proof

(a) $$\|(I+T)u\|_\tau = \|(I+T_A-E)u\|_\tau \leq \|(I+T_A)u\|_\tau + \|Eu\|_\tau$$
$$\leq \|(I+T_A)u\|_\tau + \delta\ \|(I+T)u\|_\tau;\ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.72}$$

Thus

$$\|(I+T)u\|_\tau \leq \frac{1}{1-\delta}\ \|(I+T_A)u\|_\tau;\ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.73}$$

Since T is closed-loop stable, there exists an $\epsilon>0$ such that

$$\|(I+T)^{-1}v\|_\tau \leq \frac{1}{\epsilon}\ \|v\|_\tau \tag{2.74}$$

and letting $u=(I+T)^{-1}v$, we have

$$\epsilon\ \|u\|_\tau \leq \|(I+T)u\|_\tau;\ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.75}$$

Combining (2.73) and (2.75):

$$\epsilon\ \|u\|_\tau \leq \frac{1}{1-\delta}\ \|(I+T_A)u\|_\tau;\ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+ \tag{2.76}$$

and so T_A is closed-loop stable with gain

$$\|[I+T_A]^{-1}\| \leq \frac{1}{(1-\delta)}\ \|[I+T]^{-1}\| \tag{2.77}$$

(b) T_A will be closed-loop stable if the following has finite gain:

$$[I+T_A]^{-1} = [I + T(I+E)^{-1}]^{-1} = [(I+E+T)(I+E)^{-1}]^{-1}$$
$$= (I+E)(I+E+T)^{-1}. \tag{2.78}$$

Applying theorem 2.6a, T+E is closed-loop stable and so

$$\|[I+T_A]^{-1}\| \leq (1+\|E\|)\,\frac{1}{1-\delta}\,\|[I+T]^{-1}\| < \infty. \tag{2.79}$$

(c) Since $T_A=[E+T^{-1}]^{-1} = T[ET+I]^{-1}$, we can apply the division error to conclude that T_A will be closed-loop stable if ET is stable and (2.66) holds. Letting x=Tu, we get the second result (2.67). The gain is

$$\|(I+T_A)^{-1}\| \leq \frac{1+\|ET\|}{1-\delta}\,\|[I+T]^{-1}\|. \tag{2.80}$$

(d) Since

$$T_A = (I+E)T = T + ET \tag{2.81}$$

we can apply the additive error test (theorem 2.6a) to conclude that T_A will be closed-loop stable if

$$\|ETu\|_\tau \leq \delta\,\|(I+T)u\|_\tau;\ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+. \tag{2.82}$$

Letting x=Tu, this will be true if

$$\|Ex\|_\tau \leq \delta\,\|(I+T^{-1})x\|_\tau;\ \forall x\in\mathcal{G}_3, \tau\in\mathbb{R}_+. \tag{2.83}$$

The gain is

$$\|(I+T_A)^{-1}\| \leq \frac{1}{1-\delta}\,\|(I+T)^{-1}\|. \quad \blacksquare \tag{2.84}$$

Remark 1 These robustness tests are similar to the frequency domain singular value tests used for linear systems [3]. For example, if $T(s)$ is a closed-loop stable transfer function matrix and $T_A(s)=T(s)[I+E(s)]^{-1}$, then $T_A(s)$ will be closed-loop stable if

$$\sigma_{max}[E(j\omega)] < \sigma_{min}[I+T(j\omega)]; \quad \forall\ \omega\in\mathbb{R}. \tag{2.85}$$

Compare this to theorem 2.6b. Note that, as in the performance tests of section 2.4, we must check conditions for all signals in some "large" set instead of just along the real line (2.85). We will discuss this issue in more detail in the next section.

Remark 2 The incremental versions of these robustness tests are very similar and are proved the same way. They say that if the nominal loop is incrementally closed-loop stable, and the error is incrementally bounded in a certain sense, then the actual system will be closed-loop incrementally stable.

We first present an example of how to use these tests.

Theorem 2.7 (2-norm bounds in frequency domain) Let H be a stable linear operator and $\hat{H}(j\omega)$ its Fourier transform. Then if $\|u\|_2<\infty$,

$$\|Hu\|_{2,\tau} \le \|Hu\|_2 \le \sup_{\omega} |\hat{H}(j\omega)|\ \|u\|_2, \tag{2.86}$$

and

$$\|H\|_2 = \sup_{\substack{u\in\mathcal{L} \\ \tau\in\mathbb{R}_+}} \frac{\|Hu\|_{2,\tau}}{\|u\|_{2,\tau}} \le \sup_{\omega} |\hat{H}(j\omega)|. \tag{2.87}$$

Proof Using Parseval's theorem,

$$\|Hu\|_2^2 = \int_0^\infty |\hat{H}(j\omega)\hat{u}(j\omega)|^2 \, d\omega$$
$$\leq \sup_\omega |\hat{H}(j\omega)|^2 \|u\|_2^2, \tag{2.88}$$

the rest follows from definitions. ∎

Remark In general, the bounds involving $|\hat{H}(j\omega)|$ cannot be satisfied with equality unless $|\hat{H}(j\omega)|$ is constant for all ω where $\hat{u}(j\omega)$ is nonzero.

Theorem 2.8 (Multivariable Gain and Phase Margins) Suppose we have a system that we know to be closed-loop stable and its loop operator T satisfies

$$\|[I+T]u\|_{2,\tau} \geq \epsilon; \quad \forall u \in \mathcal{L}, \tau \in \mathbb{R}_+. \tag{2.89}$$

Then the closed-loop system has the following margins, in each channel:

Gain Margin: $\left(\frac{1}{1+\epsilon}, \frac{1}{1-\epsilon}\right)$

Phase Margin: $|\theta| < \cos^{-1}\left(1 - \frac{\epsilon^2}{2}\right)$,

that is, representing the linear operator L by its Fourier Transform $\hat{L}(j\omega)$, with $\hat{L}(j\omega) = \text{diag}\,\{\ell_i(j\omega)\}$, $i=1,\ldots,m$, and for all i, either

$$\ell_i(j\omega) = \ell_i \in (\frac{1}{1+\epsilon}, \frac{1}{1-\epsilon}) \tag{2.90}$$

or

$$|\ell_i(j\omega)|=1, \quad |\text{angle } \ell_i(j\omega)| < \cos^{-1}(1-\frac{\epsilon^2}{2}), \tag{2.91}$$

then TL will be closed-loop stable. See figure 2-4.

Proof Since the L we will be contemplating is linear (gains and phases only), we can use theorem 2.7 to convert the error into the frequency domain. We use theorem 2.6b and conclude that the system will be closed-loop stable if, for $\delta<1$,

$$\|[L^{-1} - I]u\|_{2,\tau} \leq \delta \ \|[I+T]u\|_{2,\tau}; \ \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+, \tag{2.92}$$

or, by (2.89) if

$$\|[L^{-1} - I]\|_2 < \epsilon. \tag{2.93}$$

Using theorem 2.7, we require

$$\|[\hat{L}^{-1}(j\omega) - I]\| < \epsilon, \tag{2.94}$$

or for gain:

$$-\epsilon < (\ell_i^{-1} - 1) < \epsilon \tag{2.95}$$

$$\frac{1}{1-\epsilon} > \ell_i > \frac{1}{1+\epsilon}. \tag{2.96}$$

For the phase, let θ be the angle of phase shift of $\ell_i(j\omega)$. Then

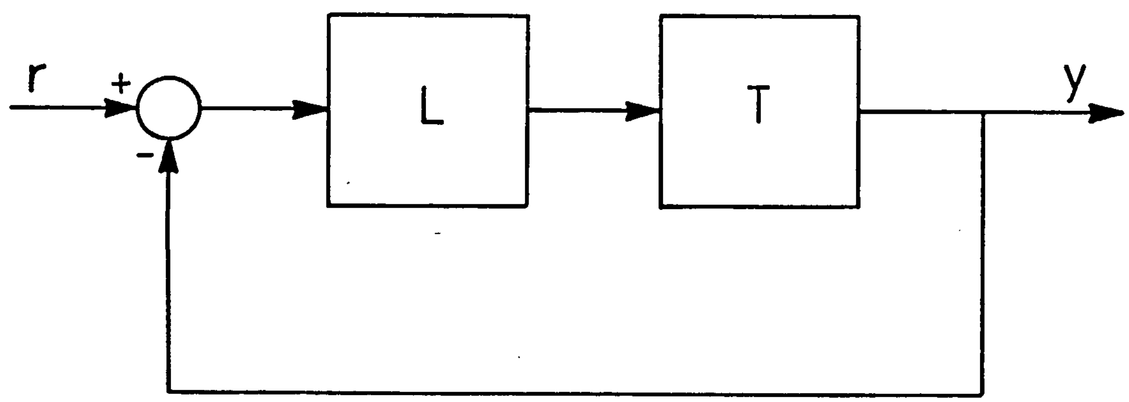


Figure 2-4: Robustness Test Application

$$\sup_{\omega} |e^{j\theta} - 1| < \epsilon \tag{2.97}$$

and

$$|\sin^2\theta + (\cos\theta - 1)^2| < \epsilon^2 \tag{2.98}$$

$$|2 - 2\cos\theta| < \epsilon^2. \quad \blacksquare \tag{2.99}$$

2.6 Computation of Tests

The actual computation of the nonlinear performance and robustness tests covered in this chapter is a rather difficult issue that will require much additional research. In this section, we will comment on a few possibilities for actully performing these tests in a practical manner.

The typical test that we are discussing is of the following form: Let the set $\mathcal{G} \subset \mathcal{L}$ be a given subset of a signal space $\mathcal{L}$. Then we must check to see if

$$\alpha(x) < \beta(x); \quad \forall\ x \in \mathcal{G}. \tag{2.100}$$

If the set $\mathcal{G}$ were more like $\mathbb{R}$, then our course of action would be clear. We would evaluate (2.100) on some smaller set, say the set

$$\{\ n\epsilon \ |\ n = -M, -M+1, \ldots, 0, 1, 2, 3, \ldots, M\ \} \tag{2.101}$$

where we select ϵ to be small enough to "cover" all the behavior of the function $\alpha(\cdot)$, and M is picked large enough to "cover" the maximum range expected to be of significance. We would be justified in doing things this way if $\alpha(\cdot)$ were smooth enough so that we did not miss any behavior that "fell through the net". The analogous situation for the (2.100) would be to find some managable, smaller set $\mathfrak{B} \subset \mathcal{G}$ that we could actually calculate (2.100) for. Then, if α were smooth enough, and the members of $\mathfrak{B}$ were close enough together, we would not miss any behavior. We would thus like to find the set $\mathfrak{B}$, together with a theorem that we could prove, saying in effect, that if (2.100) held on $\mathfrak{B}$, it would be true on all of $\mathcal{G}$. This is the principle that we use all the time for the set of real numbers, and it seems likely that it can be done, albeit carefully, for signal spaces.

One possible solution to this problem is to learn a trick from describing function theory [29,30]. In describing function theory, the set $\mathfrak{B}$ is usually taken to be a collection of sinusoids at different frequencies and amplitudes, i.e.

$$\mathfrak{B} = \{\ A\sin \omega t\ \mid A\in[-M,M],\ \omega\in[\omega_1,\omega_2]\ \} \qquad (2.102)$$

and performance and *stability* are judged on the basis of the set $\mathfrak{B}$. Now, stability is a much more complex (and delicate) issue, and while it may be possible to prove some theorems for stability of the type alluded to above, this has not been done yet. However, the performance and robustness issue may be more tractable. Perhaps one could check performance be a grid of signals, with amplitude in one direction and magnitude in the other.

Another issue along similar lines is the actual posing of specifications. Since it would be impractical to specify β in (2.120) for each $x\in G$, we need a more compact way of specifying β. Again, perhaps describing function theory could be of help. For example, we might specify performance/robustness for a control system as the sets in which A and ω must lie for $x = A\sin\omega t$:

$$\frac{\|Gx\|}{\|x\|} > 20 \qquad \omega \leq 2 \text{ rad/sec}, \; A > 20 \tag{2.103a}$$

$$\frac{\|Gx\|}{\|x\|} \geq 10 \qquad \omega \leq 2 \text{ rad/sec}, \; A \leq 20 \tag{2.103b}$$

$$\frac{\|Gx\|}{\|x\|} \leq 1 \qquad \omega > 40 \text{ rad/sec}. \tag{2.103c}$$

How one comes up with such a specification is a very complex issue in its own right which we will not address here.

Another possible way of posing specifications is to use a "chirp", or swept sine-wave signal. This type of signal, which consists of a constant amplitude sine-wave with a frequency that increases exponentially in time, might allow one to look at the time response of, say, the sensitivity operator, and interpret its envelope as a frequency response. Section 5.6 demonstates this idea on a nonlinear pendulum example.

We are now in a (theoretical) position to evaluate a given loop operator when presented with it. The next two chapters will deal with the issue of synthesis: How do we design a compensator to give a loop operator that will test "good" using the techniques presented in this chapter?

CHAPTER 3. SYNTHESIS: ESTIMATION AND CONTROL

3.1 Introduction

In chapter 2, we studied the analysis of a given feedback loop in terms of its stability, robustness, and performance. The purpose of this chapter is to present methods for actually constructing a compensator to achieve closed-loop stability for a given plant model. The issues of performance and robustness will be covered in chapter 4. This chapter is structured as follows. Section 3.2 discusses the separation principle that allows us to break the problem of stabilization into two pieces: estimation and control. Section 3.3 discusses specific estimators with emphasis on the extended Kalman filter and the guaranteed properties that it possesses making it useful for control system design. Similarly, section 3.4 discusses state feedback controllers with emphasis on the guaranteed properties possessed by the solution to certain optimal control problems, and their relevance to feedback loop properties.

3.2 Separation of Estimation and Control

The separation principle to be presented in this section allows us to separate the stabilization problem into two parts that can each be tackled separately. This provides the justification for the estimation and control approach to control system design, in which the compensator contains a state estimator and state feedback gain.

Consider the linear situation first. There we design a Kalman filter to estimate the state of a linear system given the output observations, and a state-feedback gain that would stabilize the system if the state were available for direct measurement. For example, consider

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{3.1a}$$

$$y(t) = C\,x(t) \tag{3.1b}$$

with the Kalman filter

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + H[y(t) - C\hat{x}(t)] \tag{3.2}$$

and state-feedback

$$u(t) = -\,G\,\hat{x}(t)\ . \tag{3.3}$$

One of the most fundamental results of linear system theory is that the closed-loop eigenvalues of such a system are the eigenvalues of the Kalman filter and the closed-loop eigenvalues that would result from the exact state being used instead the estimated state. Thus, the closed-loop eigenvalues are

$$\lambda_i[A-BG];\ i=1,2,\ldots,n \tag{3.4}$$

and

$$\lambda_i[A-HC];\ i=1,2,\ldots,n. \tag{3.5}$$

Thus we see that if (a) the filter produces "good" estimates, and (b) the state-feedback would stabilize the system itself, then the combination stabilizes the linear system (3.1). See figure 3-1. While the usual proof of this result relies on the assumed linearity to decouple the closed-loop equations, the result is by no means restricted to the linear case.

In order to generalize this result, we first need to define a "good" estimator. The terminology used here is due to [6,43].

Definition $\hat{x}=F(y,u)$ is a *nondivergent estimate* of the state x of

$$\dot{x}(t) = f(x(t)) + Bu(t) + Bw(t) \tag{3.6a}$$

$$y(t) = Cx(t) + d(t) \tag{3.6b}$$

if the mapping $(w,d)\mapsto e=x-\hat{x}$ is stable uniformly in u. Here F is the dynamic operator representing the estimator with inputs y and u, and w and d are disturbances that are considered deterministic (but of course unknown to the estimator). To be more precise, we say that the estimator is nondivergent with respect to a specific norm if the mapping $(w,d)\rightarrow e$ is stable with respect to that norm.

Definition If there exists a nondivergent estimator $\hat{x}=F(y,u)$ for (3.6), then we say that the system (3.6) is *detectable*.

We can now state and prove our main separation result:

Theorem 3.1 (Separation Theorem [6]) If $g(\cdot)$ is a stabilizing state-feedback function, i.e. if

$$\dot{x}(t) = f(x(t)) - Bg(x(t)) + Bw(t) \tag{3.7}$$

is stable $w\mapsto x$, and

$$\sup_{x} |\nabla g(x)| < \infty \tag{3.8}$$

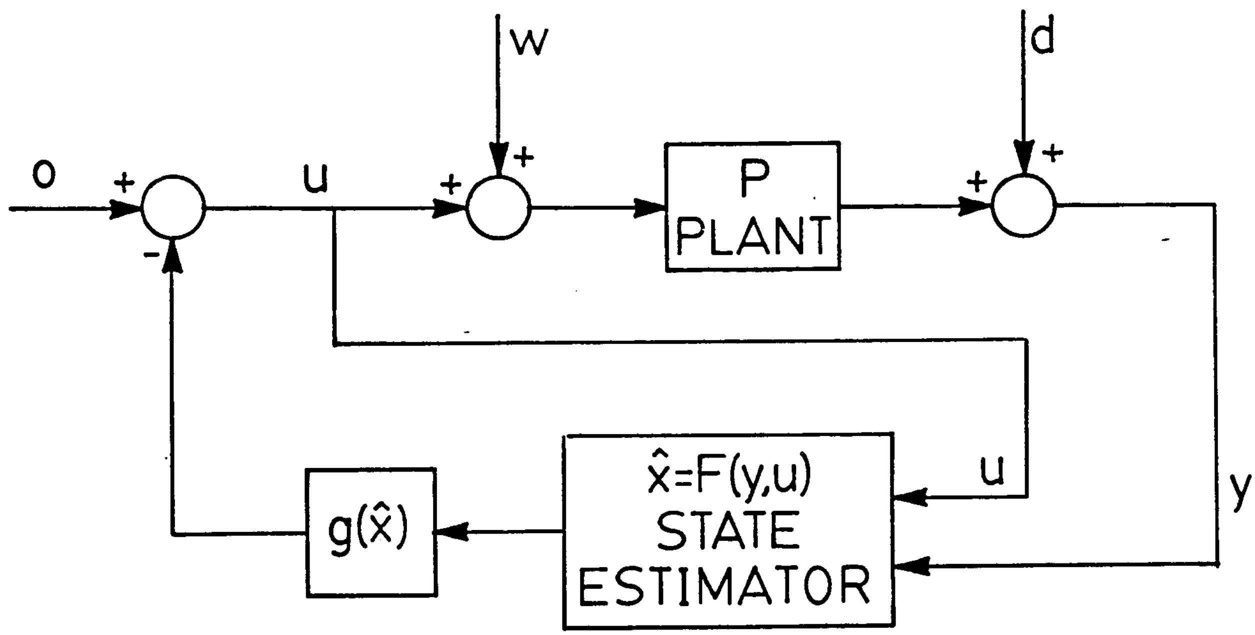


Figure 3-1: Separation of Estimation and Control

and if $\hat{x}=F(y,u)$ is any nondivergent estimate of x, then

$$\dot{x}(t) = f(x(t)) - Bg(\hat{x}(t)) + Bw(t) \tag{3.9}$$

is stable $(w,d)\mapsto x$. Here we mean stability with respect to the same norm used for the stability of (3.7) and for the nondivergence of the state estimate $\hat{x}$.

Proof The closed-loop system is

$$\dot{x} = f(x) - Bg(\hat{x}) + Bw = f(x) - Bg(x) + B(g(x)-g(\hat{x})+w). \tag{3.10}$$

Since g(x) stabilizes the system, there must exist a k_1 such that

$$\|x\|_T \leq k_1\|g(x)-g(\hat{x})+w\|_T \leq k_1|\nabla g|\cdot\|x-\hat{x}\|_T + k_1\|w\|_T;\ \forall T\in\mathbb{R}_+. \tag{3.11}$$

Since F is a nondivergent estimator, there must exist k_2 such that

$$\|x-\hat{x}\|_T \leq k_2\|(w,d)\|_T;\ \forall T\in\mathbb{R}_+ \tag{3.12}$$

and so

$$\|x\|_T \leq k_1|\nabla g|k_2\|(w,d)\|_T + k_1\|w\|_T;\ \forall T\in\mathbb{R}_+, \tag{3.13}$$

and so the system is closed-loop stable. ∎

Remark 1 This theorem now allows us to design separately a stabilizing state-feedback function and a nondivergent estimator, with the knowledge that we can put them together and be guaranteed a closed-loop stable system. Note that the stability is not just from a single input, but from both "inputs" (w,d) simultaneously. This guarantees that there will be no unstable hidden modes in the closed-loop system, i.e. it rules out the analog of right-half plane pole-zero cancellations between the compensator and plant in linear systems. This is required (and sufficient) to allow a practical command following system to work. This is the approach that will be taken in this thesis.

Remark 2 In the linear case, the stochastic optimal control (Linear-Quadratic-Guassian, or LQG) problem solution [50] decouples into an optimal estimation problem and an optimal state-feedback control problem, sometimes refered to as the certainty equivalence property. We do not mean to imply that the nonlinear stochastic optimal control problem [53] has a similar property; only that we can *stabilize* nonlinear systems by this separation process.

Remark 3 In the literature, there exist many tests for stability of a closed-loop system [4,5,6,7, and many others]. All of these are based on versions of the small-gain theorem and/or passivity theorems. The problem with any of these tests is that they require that either one or both of the compensator and plant must be open-loop stable. Since there are some linear systems which cannot be stabilized with a stable compensator, we would expect the same to be true for some nonlinear systems. Thus these tests would beuseless in trying to determine the closed-loop stability of a proposed compensator for such a plant. The separation theorem above has no such problem. It works equally well on open-loop unstable plants and compensators. Thus it could almost be viewed as a type of stability test fundamentally different from pre-existing ones of the small-gain or passivity type.

Remark 4 If the condition (3.8) is not satisfied globally, we can still make a small-signal version of the conclusion. Equation (3.8) should hold (if g is smooth) in any bounded subset of $\mathbb{R}^n$, and thus if we put the correct bounds on the size of the inputs w,r,d, we can make sure that $x, \hat{x}$ remain in that bounded subset. This allows us to guarantee closed-loop stability for inputs with magnitudes below some specific value.

We now present additional results on estimation.

Theorem 3.2 Let $\hat{x}=F(y,u)$ be an estimate for (3.6). Then the following are equivalent:

(a) $\hat{x}$ is a nondivergent estimate for x.

(b) $x = F(y-d,u+w)$, and F is incrementally stable in the special sense that

$$\|F(y-d,u+w)-F(y,u)\|_T \le \|(w,d)\|_T; \quad \forall T\in\mathbb{R}_+,\ u,w,d\in\mathcal{L} \qquad (3.14)$$

where y is determined by the choice of $u,w,d\in\mathcal{L}$.

Proof (a) implies (b):

From inspection of figure 3-1, y-d is the signal at the output of the plant and u+w is the signal at the input to the plant. Thus, if the estimator acts on (y-d,u+w), it will act as if there were *no noises* and must produce zero estimation error, as it is nondivergent:

$$\|x-\hat{x}\|_T = \leq k \, \|(w,d)\|_T; \;\; \forall \tau \in \mathbb{R}_+, w, d \in \mathcal{L}. \tag{3.15}$$

Therefore, we have

$$\|x-\hat{x}\|_T = \|F(y-d,u+w)-F(y,u)\|_T \leq k\|(w,d)\|_T; \;\; \forall \tau \in \mathbb{R}_+, u,w,d \in \mathcal{L} \tag{3.16}$$

where y is determined by $u,w,d \in \mathcal{L}$.

(b) implies (a):

$$\|e\|_T \leq \|x-\hat{x}\|_T = \|F(y-d,u+w)-F(y,u)\|_T \leq k \, \|(w,d)\|_T; \;\; \forall \tau \in \mathbb{R}_+ \tag{3.17}$$

for all $w,d,u \in \mathcal{L}$ because F is incrementally stable. ∎

Remark The statement (b) of theorem 3.2 cannot be the standard definition of incremental stability. This is because we cannot quite exercise all the signals, as y is determined by choice of u,w, and d. Note that it is incrementally stable if we consider only the second argument of F (the u, u+w terms), or, equivalently, fix d and vary w.

Theorem 3.3 Let $\hat{x}=F(y,u)$ be a nondivergent estimator for (3.6) with constant k in (3.15). Then

$$\|F(y,u)\|_T \;\leq\; k\,\|(y,u)\|_T; \quad \forall\; y,u\in\mathcal{L},\tau\in\mathbb{R}_+. \tag{3.18}$$

Proof Let w=−u, so that x=0 and y=d. Then

$$\begin{aligned}\|e\|_T &= \|x-\hat{x}\|_T = \|F(y-d,u+w)-F(y,u)\|_T = \|F(0,0)-F(y,u)\|_T \\ &= \|F(y,u)\|_T \;\leq\; k\,\|(y,u)\|_T; \quad \forall\; \tau\in\mathbb{R}_+,\end{aligned} \tag{3.19}$$

by nondivergence. ∎

Theorem 3.4 If a system described by (3.6) is detectable, then that system is in $\mathcal{SD}$.

Proof Let (w,d)=0. Then the estimator produces exact estimates, and

$$\|x\|_T = \|\hat{x}\|_T = \|F(y,u)\|_T \leq k\,\|(y,u)\|_T \leq k_1\|y\|_T + k_2\|u\|_T, \tag{3.20}$$

where the constants k_1,k_2 exist by the definition of $\|(\cdot,\cdot)\|$. ∎

Theorem 3.5 If (3.6) is incrementally stable from $u\mapsto x$, then it is detectable and thus there exists a nondivergent estimator for it.

Proof Simply use as an estimator $\hat{x}=\Phi Bu$. Then

$$\|e\|_T = \|x-\hat{x}\|_T \leq \|\Phi B\|_\Delta \|w\|_T \tag{3.21}$$

which makes the estimate nondivergent. ∎

Remark Now we see that we can actually use the internal stability results of section 2.3.2, as now can present a class of systems that are all in $\mathcal{SD}$: the set of detectable systems.

3.3 State Estimators

3.3.1 Introduction

The separation theorem justifies a search for a nondivergent estimator, as it guarantees that we will be able to use it in the final closed-loop system. In this section we proceed to examine specific state estimators. We start in section 3.3.2 by examining optimal estimation algorithms. Although they might be guaranteed nondivergent, we show that they have some rather severe computational requirements, in general. Even with the astounding computational advances of the past decade, we still cannot do optimal estimation in real time for reasonable problems (except for linear systems). Therefore we look to other, more approximate, estimators and observers. In section 3.3.3 and 3.3.4, we examine the extended Kalman filter and show that it has some remarkable guaranteed properties, including nondivergence. We continue the examination by looking at simpler, constant linear gain observers in section 3.3.5 and a special form of observer that we shall call the *transformation based observer* in section 3.3.6. Related material appears in appendices C and D. Appendix C discusses a promising new observer and appendix D shows the guaranteed properties of the optimal nonlinear estimator.

3.3.2 Optimal Nonlinear Estimation and Its Problems

For comparision purposes, we consider the optimal deterministic estimator at first. For review, the goal is as follows. For the system (3.6), find a dynamic system $\hat{x}=F(y,u)$ that is nondivergent, i.e.

$$\|e\|_\tau = \|x-\hat{x}\|_\tau \le k \ \|(w,d)\|_\tau; \ \forall \tau \in \mathbb{R}_+, \ w,d,u \in \mathcal{L}. \tag{3.22}$$

One way of posing an optimal estimation problem is to use a minimum energy approach [34,35]. In this approach, one selects the estimate $\hat{x}(T)$ as the endpoint of the minimum energy trajectory, i.e. the one that minimizes

$$J = \int_0^T [\,|y-Cz|^2 + q|w|^2\,]dt \tag{3.23}$$

where

$$\dot{z} = f(z) + Bu + Bw \tag{3.24}$$

$$\hat{x}(T) = z(T). \tag{3.25}$$

The solution to this optimization problem is as follows:

$$\hat{x}(T) = \underset{x}{\operatorname{argmin}}\ V(x,T) \tag{3.26}$$

where $V(x,t)>0$ satisfies the partial differential equation

$$V_t = [y(t)-Cx]^T[y(t)-Cx] - V_x{}'f(x) - \tfrac{1}{2}V_x{}^T \Xi V_x \tag{3.27}$$

where $\Xi=\Xi^T \ge 0$ and

$$V_x := \left[\frac{\partial V}{\partial x_1} \ \frac{\partial V}{\partial x_2} \ \cdots \ \frac{\partial V}{\partial x_n} \right]^T \tag{3.28}$$

$$V(x,0) = V_o(x). \tag{3.29}$$

V_o is the initial condition for the partial differential equation and corresponds to the a priori probability density in a stochastic problem. The estimate $\hat{x}$ given by (3.26) evolves according to

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + Bu(t) + V_{xx}^{-1}(\hat{x},t)[y(t)-C\hat{x}(t)] \tag{3.30}$$

where V_{xx} is the matrix of second partials of V.

In appendix D, we show that the optimal nonlinear filter has a guaranteed return difference condition property. The reader may wish to postpone refering to appendix D until after sections 3.3.4 and 3.4.3 which detail similar properties for the extended Kalman filter and the optimal nonlinear regulator.

The problem with the optimal nonlinear estimator presented above is that it is infinite-dimensional. To compute it, one must actually compute the solution to the partial differential equation (3.27) in real time. An actual implementation would require the storage and update of V(x,t) at each discretization interval. Consider just the storage requirements for V(x,t) for some fixed t. If we discretize the state space into just, say, 100 segments in each dimension, we see that the storage requirements for a system of only 4 states is $100^4 = 10^8$. This is an incredible amount of storage, and we have not yet even considered that we must update each of these locations at each discretization interval in time. Thus we must conclude, that for even moderately complex systems, with the time scales that we tend to see in typical control applications, this is far too much storage and computation.

In the linear case, we have several simplifying phenomena. The partial differenial equation turns out to be finite-dimensional, and V_{xx} turns out to be independent of the measurements. As we let time go to infinity, V_{xx}^{-1} approaches the steady-state covariance of the system, and we get the steady-state Kalman filter, with its even further reduced computational burden (the covariance is constant in time).

Remark 1 If we had a relatively slow system of not too high order, we might be able to calculate the optimal state estimate. Such systems might appear in the process control industry, however in the case of such slow systems, other types of estimation might be more suitable.

Remark 2 There are many other formulations of the optimal filtering problem in the stochastic case, depending on the exact cost functional chosen. For example, see [36,37,38,39]. Unfortunately, they all have the drawback of requiring the solution in real time of a partial differential equation with no useful steady-state solution.

Remark 3 In the literature, there are some examples of finite-dimensional exact estimators. These are special cases of the estimator discussed above, in which the filter just happens to be finite-dimensional. These cases seem to be rather restrictive. In fact, it can be proven that, in general (i.e. generically), the optimal nonlinear estimator is infinite-dimensional [40].

The above rather unpleasant state of affairs motivates us to look beyond optimal estimation to some type of approximate estimator. There are a great many approximate estimators or observers in the literature [35,37,41,42,43,44,45]. Some of these have conditions that can be checked to ensure the nondivergence of the state estimate. For example, [41,43] involve checking the uniform positive definiteness of an operator or matrix over a vector space (see section 3.3.5), while [44] involves checking the boundedness of the solution of a specific matrix differential equation. However, none of these aprroximate estimators have any documented guaranteed properties (i.e. ones that can be verified easily a priori), such as nondivergence, that we would like in order to use them for control system design. Fortunately, there do exist nonlinear approximate filters with guaranteed properties. We present two of these in this thesis, namely the *costate observer* and the *extended Kalman filter*. Appendix C presents the results of a preliminary investigation into a new type of filter, the costate observer. In the next two sections, we present the results of research on the extended Kalman filter (EKF) and its guaranteed properties, including its nondivergence.

3.3.3 The Extended Kalman Filter and Nondivergence

The extended Kalman filter (EKF) was proposed as an engineering extension to the popular Kalman Filter for linear systems [35]. Later references are [39,46]. The EKF as we will use it for the nonlinear system (3.6) is

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + Bu(t) + H(t)[y(t)-C\hat{x}(t)]; \quad \hat{x}(0)=x_o \tag{3.31}$$

$$H(t) = \Sigma(t)C^T \tag{3.32}$$

$$\dot{\Sigma}(t) = \nabla f(\hat{x}(t))\Sigma(t) + \Sigma(t)\{\nabla f(\hat{x}(t))\}^T + \Xi - \Sigma(t)C^T C\Sigma(t) \tag{3.33}$$

$$\Sigma(t_o) = \Sigma_o; \; t_o<0. \tag{3.34}$$

Here, the symmetric and at least positive semidefinite matrix Ξ is one of the design parameters of the EKF. We shall frequently refer to the square-root of Ξ, written $\Xi^{1/2}$, defined as the full-rank matrix Γ such that

$$\Gamma\Gamma^T = \Xi. \tag{3.35}$$

The other parameters of the EKF are the initial time $t_o<0$ and the initial state for the covariance propogation equation (3.33). The results reported here will require a "start-up" period for the EKF if it is to be initialized with arbitrary Σ_o; that is, we must have $t_o<c$ for some $c<0$ and (3.31) starts at $t=0$. Obviously, we could start the EKF at $t_o=0$ if we selected an appropriate Σ_o. This is the procedure that would be used in practice.

The rationale for the EKF was that if the noises were small enough, $x\approx\hat{x}$, and one would be justified in using the standard time-varying Kalman filter because (3.33) would then be a good approximation of the true error covariance. It turned out that the EKF was very good in practice and many papers and applications were reported of the EKF and its variants, including [47,48]. As we shall show, this was not just pure chance, but a consequence of certain guaranteed properties possessed by the EKF.

We start by making some definitions based on [49].

Definition We say that $[A(\cdot),C(\cdot)]$ is *uniformly observable* if for the linear time varying (LTV) system

$$\dot{\xi}(t) = A(t)\xi(t); \quad \xi(0)=\xi_0 \tag{3.36a}$$

$$y(t) = C(t)\xi(t) \tag{3.36b}$$

there exist constants α,β,σ such that the observability grammian

$$W(t_0,t_1) := \int_{t_0}^{t_1} \Phi^T(s,t_1)C^T(t)C(t)\Phi(s,t_1)ds \tag{3.37}$$

is bounded uniformly

$$\beta I > W(t_0,t_0+\sigma) > \alpha I > 0 \tag{3.38}$$

for all $t_0 \in \mathbb{R}_+$. Here Φ is the state transition matrix for the linear system (3.36a). Similarly, we say that $[A(\cdot),B(\cdot)]$ is *uniformly controllable* if for the linear time-varying system

$$\dot{\xi}(t) = A(t)\xi(t) + B(t)u(t); \quad \xi(0)=\xi_0 \tag{3.39}$$

there exist α,β,σ such that the controllability grammian

$$C(t_0,t_1) := \int_{t_0}^{t_1} \Phi(t_1,s)B(t)B^T(t)\Phi^T(t_1,s)ds \tag{3.40}$$

is bounded uniformly

$$\beta I > C(t_0,t_0+\sigma) > \alpha I > 0 \tag{3.41}$$

for all $t_0 \in \mathbb{R}_+$.

Remark If we make the further assumption that $A(t) \leq M$ for some constant M, then the upper bounds in (3.38) and (3.41) are satisfied automatically. Recall that for constant linear systems, the crucial part of observability and controllability are the lower bounds, i.e. the positive definiteness of the grammians.

Definition A nonlinear system $[f,C]$ of the form (3.6) is *L-observable* (for Linearization observable), if uniformly for every possible trajectory $x \in \mathcal{L}$, the linearized system $[\nabla f(x(\cdot)),C(\cdot)]$ is uniformly observable. Similarly, the nonlinear system $[f,B]$ is *L-controllable* (for Linearization controllable) if $[\nabla f(x(\cdot)),\Xi^{1/2}]$ is uniformly controllable, uniform across all trajectories $x(\cdot)$. The uniformity across trajectories here means that the bounds α,β in the definitions of uniform observability and controllability are the same for all $x(\cdot) \in \mathcal{L}$.

Definition A nonlinear system $[f,C]$ of the form (3.6) is *M-detectable* (for Model-based detectable) if there exists a matrix functional $H(t,y(s),u(s),0 \leq s < t)$, depending on the past of y and u, such that for any matrix B in our plant model (3.6), the state estimate given by

$$\dot{\hat{x}}(t) = f(\hat{x}) + Bu(t) + H(t,y(s),u(s),0 \leq s < t)[y(t)-C\hat{x}(t)] \qquad (3.42)$$

is nondivergent, uniformly for all matrices B, i.e. for all $B \in \mathbb{R}^{n \times p}$ and for all p. In addition, the functional H must be bounded in time, and continuous, not necessarily uniformly, with respect to $y(\cdot)$. This means that given a $\epsilon, \tau > 0$, there exists a $\eta(\epsilon,\tau)$ such that if

$$\|y_1 - y_2\|_{\infty,\tau} \leq \eta(\epsilon,\tau) \qquad (3.43)$$

then

$$|H(t,y_1(s),u(s),\ 0 \leq s \leq t)C - H(t,y_2(s),u(s),0 \leq s \leq t)C| \leq \epsilon; \ \forall\ 0 \leq t \leq \tau. \quad (3.44)$$

Remark . The matrix function $H(\cdot)$ in (3.42) can depend in any way on the past of u and y. Thus it includes the optimal infinite-dimensional observer discussed in section 3.3.1, as well as the EKF, and a host of other approximate observers. Additionally, the observer (3.42) must be nondivergent independent of B. This is in keeping with the linear theory, where choice of B matrix does not influence observability. Thus, M-detectability is one of the most fundamental definitions for detectability that one can make, since it is operational in nature: If the system is not M-detectable then we cannot find an estimator that will be nondivergent for all choices of the B matrix. In this sense it is analogous to detectability in linear system theory.

We now state our results pertaining to the EKF:

Theorem 3.6 Let f obey the gradient restriction (2.2). Then if

$$\sigma_{\min}[\Xi + \Sigma(t)C^T C\Sigma(t)] \geq \epsilon > 0; \ \forall t\in\mathbb{R}_+, \tag{3.45}$$

and one of the following holds:

(a) $[f,C]$ is M-detectable and $[f,\Xi^{1/2}]$ is L-controllable.

(b) $[f,C]$ is L-observable and $[f,\Xi^{1/2}]$ is L-controllable,

(c) $\Sigma(t)$ is bounded in time, i.e. there exist $\alpha,\beta>0$ such that

$$\beta I \geq \Sigma(t) \geq \alpha I > 0; \ \forall t\in\mathbb{R}_+, \tag{3.46}$$

then the EKF (3.31-3.34) is a nondivergent estimator for the nonlinear system (3.6). Furthermore, (a) implies (c), and (b) implies (c).

Proof See Appendix B. The steps in the proof can be read for a sketch if the reader is not interested in the details.

Remark 1 This is a very useful theorem, as it says that if any nondivergent estimator exists, then the EKF will also work for control purposes. Note that this nondivergenbce is global, as it says nothing about the noises w,d being small. Note further that the condition (3.45) can be easily satisfied by picking Ξ positive definite, as can the condition for $[f,\Xi^{1/2}]$ being L-controllable. When Ξ is positive semi-definite the conditions (3.45) and $[f,\Xi^{1/2}]$ controllable are more difficult to check. It would seem that it should only require some form of stabilizability for $[f, \Xi^{1/2}]$, where we would require the existence of a stabilizing state feedback function, but at this time this is not known.

Remark 2 One should be able to prove a stochastic version of this theorem, perhaps by using a norm $\|x\|$ that was related to the covariance of $x(t)$. In addition, due to the connection of the EKF with the linear Kalman filter, one would also expect some result saying, in effect, that no other filter has a better local estimation error covariance.

Remark 3 If one were optimistic, one would be tempted to draw the conclusion that a dual result to this EKF nondivergence result could be made, that is, using some form of the time-varying Linear-Quadratic regulator problem [50], one could derive guaranteed stable state feedback functions without having to solve partial differential equations. Unfortunately, this cannot work, as the control matrix Ricatti equation must be propagated backwards in time, and we do not know what our linearized trajectory will be at any time in the future. We are lucky in the filtering case, as the Kalman filter runs forward in time, and we do not need to know $A(t)$ for any time in the future.

3.3.4 Guaranteed Properties of the Extended Kalman Filter

The last section proved one very important property of the EKF, its nondivergence under very general conditions. This allows us to use the EKF for a control system, as the separation theorem guarantees closed-loop stability. However, the EKF has a number of additional important properties, which we present here. As way of an introduction, we start by giving a useful result for the linear time-varying Kalman filter.

Theorem 3.7 For the linear time-varying system and Kalman Filter

$$\dot{\xi}(t) = A(t)\xi(t) + H(t)u(t) \tag{3.47}$$

$$H(t) = \Sigma(t)C^T \tag{3.48}$$

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A(t) + \Xi - \Sigma(t)C^TC\Sigma(t), \tag{3.49}$$

$$\Sigma(t_o)=\Sigma_o, \tag{3.50}$$

the following hold:

$$\|[I+C\Phi H]u\|_{2,\tau} \geq \|u\|_{2,\tau} \; ; \quad \forall u\in\mathcal{L}, \; \tau\in\mathbb{R}_+. \tag{3.51}$$

$$\|[I+C\Phi H]^{-1}v\|_{2,\tau} \leq \|v\|_{2,\tau} \; ; \quad \forall v\in\mathcal{L}, \; \tau\in\mathbb{R}_+ \tag{3.52}$$

$$\|[I+(C\Phi H)^{-1}]u\|_{2,\tau} \geq \frac{1}{2} \|u\|_{2,\tau} \; ; \quad \forall u\in\mathcal{L}, \; \tau\in\mathbb{R}_+, \tag{3.53}$$

where $C\Phi H$ is the operator mapping $u\mapsto C\xi$ in (3.47-3.50).

Proof Let

$$P(t) := \Sigma^{-1}(t) \tag{3.54}$$

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) - C^TC + P(t)\Xi P(t) \tag{3.55}$$

$$v(\xi,t) = \frac{1}{2} \xi^T P(t)\xi, \tag{3.56}$$

$$\frac{dv(\xi(t),t)}{dt} = \frac{1}{2} \xi^T(t)\dot{P}(t)\xi(t) + \xi^T(t)P(t)[A(t)\xi(t)+\Sigma(t)C^Tu(t)] \tag{3.57}$$

and dropping the t argument,

$$\int_o^T [u^T u + \xi^T P \Xi P \xi] dt = \int_o^T \left\{ [u+C\xi]^T [u+C\xi] + \xi^T \{-\dot{P}-PA-A^T P\} \xi - 2u^T C\xi \right\} d\tau$$

$$= \int_o^T \left\{ [u+C\xi]^T [u+C\xi] - 2\dot{v} \right\} d\tau$$

$$= \int_o^T \left\{ [u+C\xi]^T [u+C\xi] \right\} d\tau - [v(\xi(\tau),\tau) - v(\xi(0),0)]. \qquad (3.58)$$

Since $\xi(0)=0$, and $v \geq 0$ we are nearly finished. We simply identify the y with the operator notation:

$$y = C\Phi H\ u, \qquad (3.59)$$

and

$$v = [I+C\Phi H]\ u. \qquad (3.60)$$

The result (3.53) follows by simple algebra. ∎

Remark This the analog of the linear time-invariant results [3]

$$\sigma_{min}[I+C\Phi(s)H] \geq 1; \quad s=j\omega, \qquad (3.61)$$

$$\sigma_{max}[\{I+C\Phi(s)H\}^{-1}] \leq 1; \quad s=j\omega, \qquad (3.62)$$

$$\sigma_{max}[I+(C\Phi(s)H)^{-1}] \leq \frac{1}{2}; \quad s=j\omega. \qquad (3.63)$$

These results have important implications for the robustness of the loop $C\Phi H$, and we shall see that similar results hold for the nonlinear EKF.

Theorem 3.8 Let the EKF be a nondivergent estimator by theorem 3.6. Then the system x=Gu, described by

$$\dot{x}(t) = f(x(t)) - H(t)Cx(t) + u(t); \; x(t_0)=0 \tag{3.64}$$

is L_p-incrementally stable $1 \le p \le \infty$, where $H(t)$ is the EKF gain based on the linearization $A(t)=\nabla f(x(t))$.

Proof For L_p-stability, we use theorem 2.3 and the proof of the EKF nondivergence (where we show that (3.64) is exponentially stable) to show: for $\|w\|_{\infty,\tau} \le \epsilon$,

$$\|Gu - G(u+w)\|_{p,\tau} \le k_2\|w\|_{p,\tau} \;\; ; \; \forall\tau\in\mathbb{R}_+. \tag{3.65}$$

We then piece together the whole space $\mathcal{L}$ by the same trick as in theorem 2.4. Fix $\tau\in\mathbb{R}_+$, let $w\in\mathcal{L}$ be arbitrary, and $r=\|w\|_{\infty,\tau}$. Let n be an integer such that

$$n > \frac{r}{\epsilon} \; . \tag{3.66}$$

Then

$$\|Gu-G(u+w)\|_{p,\tau}$$

$$= \left\|Gu-G(u+w\tfrac{1}{n})+G(u+w\tfrac{1}{n})-G(u+w\tfrac{2}{n})+G(u+w\tfrac{2}{n})-\dots -G(u+w\tfrac{n-1}{n})+G(u+w\tfrac{n-1}{n})-G(u+w\tfrac{n}{n})\right\|_{p,\tau}$$

$$\le \left\|Gu-G(u+w\tfrac{1}{n})\right\|_{p,\tau}+\left\|G(u+w\tfrac{1}{n})-G(u+w\tfrac{2}{n})\right\|_{p,\tau}+\dots \left\|G(u+w\tfrac{n-1}{n})-G(u+w\tfrac{n}{n})\right\|_{p,\tau}$$

$$\le \frac{1}{n}k_2\|w\|_{p,\tau}+\dots+\frac{1}{n}k_2\|w\|_{p,\tau}$$

$$= k_2\|w\|_{p,\tau} \tag{3.67}$$

because

$$\|\tfrac{1}{n}\cdot w\|_{\infty,\tau} = \frac{r}{n} \le \epsilon \tag{3.68}$$

is small enough to allow us to use the small-signal result. ∎

In general, we will use $C\Phi H$ to indicate the operator where H is determined through the EKF equations (3.31-3.34) where $\hat{x}$ is the state internal to the Φ operator. Thus, $C\Phi H[I+C\Phi H]^{-1}$ is given by the map $v \mapsto Cx$ in (3.64), where $u(t)=H(t)v(t)$.

Corollary 3.9 For the EKF, $[I+C\Phi H]^{-1}$ is L_p-incrementally stable, $1 \le p \le \infty$.

Proof The map $P_1=[I+C\Phi H]^{-1}$ is given by $u\mapsto u-CPH(t)u$ in (3.64) above.

$$\|P_1u-P_1(u+w)\|_{p,\tau} = \|w + C[PH(t)u-PH(t)(u+w)]\|_{p,\tau}$$
$$\leq [1+|C|k]\ |\Sigma(t)C|\ \|w\|_{p,\tau};\ \forall w\in\mathcal{L},\ \forall\tau\in\mathbb{R}_+, \tag{3.69}$$

where $\Sigma(t)$ must be bounded. ∎

Theorem 3.10 For $i=1,2$, let $u_i\in\mathcal{L}$, and let x_i be the corresponding trajectory through the EKF loop equation (3.64), and H_i be the corresponding EKF gain trajectory. Let k be the L_2-gain of the EKF loop in theorem 3.8. Assume that

$$\|u\|_\infty \leq M \tag{3.70}$$
$$|H_1(t) - H_2(t)| \leq \delta\ \|u_1-u_2\|_\infty;\ \forall t>t_0. \tag{3.71}$$

Then if $k\delta M<1$, then

$$\|[I+C\Phi H]^{-1}u_1-[I+C\Phi H]^{-1}u_2\|_{2,\tau} \leq \left[\ 1\ +\ |C|\frac{k^2\delta M}{1-k\delta M}\right] \|u_1-u_2\|_{2,\tau} \tag{3.72}$$

Proof Define the trajectories:

$$\dot{x}_1 = f(x_1) - H_1(t)Cx_1 + u_1;\quad y_1=u_1-Cx_1;\quad x_1(0)=0, \tag{3.73}$$
$$\dot{x}_2 = f(x_2) - H_2(t)Cx_2 + u_2;\quad y_2=u_2-Cx_2;\quad x_2(0)=0, \tag{3.74}$$
$$\dot{x}_3 = f(x_3) - H_1(t)Cx_3 + u_2;\quad y_3=u_2-Cx_3;\quad x_3(0)=0. \tag{3.75}$$

Then we have

$$
\begin{aligned}
&\|[I+C\Phi H]^{-1}u_1-[I+C\Phi H]^{-1}u_2\|_{2,T} \\
&= \|y_1 - y_2\|_{2,T} = \|y_1 - y_3 + y_3 - y_2\|_{2,T} \\
&\leq \|y_1 - y_3\|_{2,T} + \|y_3 - y_2\|_{2,T}.
\end{aligned} \tag{3.76}
$$

To evaluate the first term, we can use theorem 2.4 and theorem 3.7, because the systems (3.73) and (3.75) are time-varying in the sense of theorem 2.4, as the gain H is the same in both ($H_1(t)$). The second term we can do the same way, as

$$\dot{x}_3 = f(x_3) - H_2(t)Cx_3 + u_2 + [H_2(t)-H_1(t)]C(x_3-x_2+x_2) \tag{3.77}$$

and so by comparing (3.74) and (3.77),

$$\|x_2-x_3\|_{2,T} \leq k\ \delta\ \|u_2-u_1\|_{2,T}\ \{\ \|x_3-x_2\|_{2,T} + \|x_2\|_{2,T}\ \} \tag{3.78}$$

and letting $r:=\|u_2-u_1\|_{2,T}$

$$\|x_2-x_3\|_{2,T} \leq \frac{k\delta r}{1-k\delta r}\ \|x_2\|_{2,T} \leq \frac{k^2\delta r}{1-k\delta M}\ \|u_2\|_{2,T} \leq \frac{k^2\delta M}{1-k\delta M}\|u_2-u_1\|_{2,T}\ . \tag{3.79}$$

we also have

$$\|y_1-y_3\|_{2,T} \leq \|u_1-u_2\|_{2,T} \tag{3.80}$$

by theorem 3.7. Thus, putting it all together

$$\|y_1-y_2\|_{2,\tau} \le \left[1 + |C|\frac{k^2\delta M}{1-k\delta M} \right] \|u_1-u_2\|_{2,\tau}. \quad \blacksquare \tag{3.81}$$

Remark This result essentially says that if the EKF gain does not vary too much from one input to the next, that the loop operator CΦH is a very robust loop. Consult figure 3-2. We can see this by referring to the robustness theorem 2.6. The main difficulty with this result is that the trajectory H(t) of the Kalman gain varies with the input and this affects the linearized trajectory. We want the linearized trajectory to be the Kalman filter, but it is not quite due to the term $\partial H/\partial u$. Thus we need the constraint (3.71) to bound this effect. Note that the variation of H with inputs is a function of the parameter Ξ. If $\Xi=q^2BB^T$, then we know that $H/q \to BW$ for some orthonormal matrix W, and thus for large q, we should reduce the constant δ in (3.71) towards zero. For convenience, we now summarize the results on the EKF.

Theorem 3.11 (Summary of EKF properties) If the EKF is nondivergent, then, assuming the constraints of (3.70-71) and letting

$$\epsilon := |C|\frac{k^2\delta M}{1-k\delta M} \tag{3.82}$$

$$\gamma := \frac{1}{1+\epsilon} \tag{3.83}$$

the following hold with respect to the loop operator shown in figure 3-2:

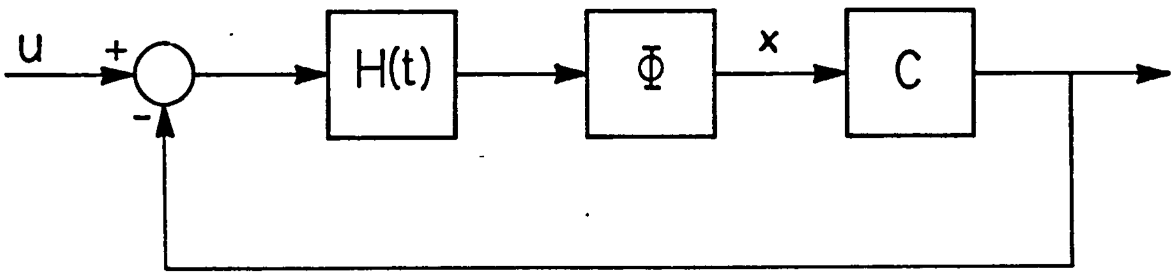


Figure 3-2: The Filter Loop: CΦH

(a) $$\|(I+C\Phi H)^{-1}\|_{2,\tau} \lesssim 1 + \gamma \tag{3.84}$$

(b) $$\|C\Phi H(I+C\Phi H)^{-1}\|_{2,\tau} \lesssim 2 + \gamma \tag{3.85}$$

(c) $$\frac{\|(I+C\Phi H)u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq \frac{1}{1+\epsilon} = \gamma \tag{3.86}$$

(d) $$\frac{\|[I+(C\Phi H)^{-1}]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq \frac{1}{2+\epsilon} \tag{3.87}$$

(e) The closed loop system of Figure 3-2 has a gain margin of

$$GM = \left[\frac{1}{1+\gamma} , \frac{1}{1-\gamma} \right] \tag{3.88}$$

and a phase margin of

$$PM = +/- \cos^{-1} \left[1 - \frac{\gamma^2}{2} \right], \tag{3.89}$$

independently and simultaneously in all channels.

Proof The only facts we have not proved as yet are (b) and (d).

(b) $$\|C\Phi H[I+C\Phi H]^{-1}\| = \|I - [I+C\Phi H]^{-1}\| \leq 1 + 1 = \gamma. \tag{3.90}$$

(d) $$\| \{I+[C\Phi H]^{-1}\}^{-1} \| = \| \{[C\Phi H+I][C\Phi H]^{-1}\}^{-1} \|$$

$$= \| C\Phi H[I+C\Phi H]^{-1} \| \leq 2 + \gamma. \quad ▮ \tag{3.91}$$

We have shown in these last two sections that the extended Kalman filter possesses some remarkable properties. We have shown that the EKF is nondivergent for detectable plants (theorem 3.6), and that the filter loop $C\Phi H$ has some good robustness properties. We shall make use of these guaranteed properties when we come to the NMBC/LQR methodology in section 4.3.

3.3.5 Constant Gain Model Based Observers

The EKF studied in the last two sections has some very desirable guaranteed properties. We would like to use it for control applications all of the time, but it has one drawback: the computational burden associated with propagating the covariance matrix may be prohibitive in certain circumstances. Thus there has been considerable interest in perhaps choosing a constant linear gain H to replace the time-varying EKF gain. We now present some results concerning the choice of H guaranteeing nondivergent estimation for the nonlinear system (3.6), as shown in figure 3-3. The following is modified slightly from [41].

Theorem 3.12. Let the constant matrix $P=P^T>0$ and constant $\alpha>0$ be such that

$$P^T[\nabla f(x) - HC + \alpha I] \leq 0; \quad \forall x \in \mathbb{R}^n, \tag{3.92}$$

uniformly. Then the gain H will produce a nondivergent estimator.

Theorem 3.13 [6] Let the constant matrices $P=P^T>0$, $S=S^T>0$, and A satisfy the Lyapunov equation

$$[A-HC]P + P[A-HC]^T + S = 0. \tag{3.93}$$

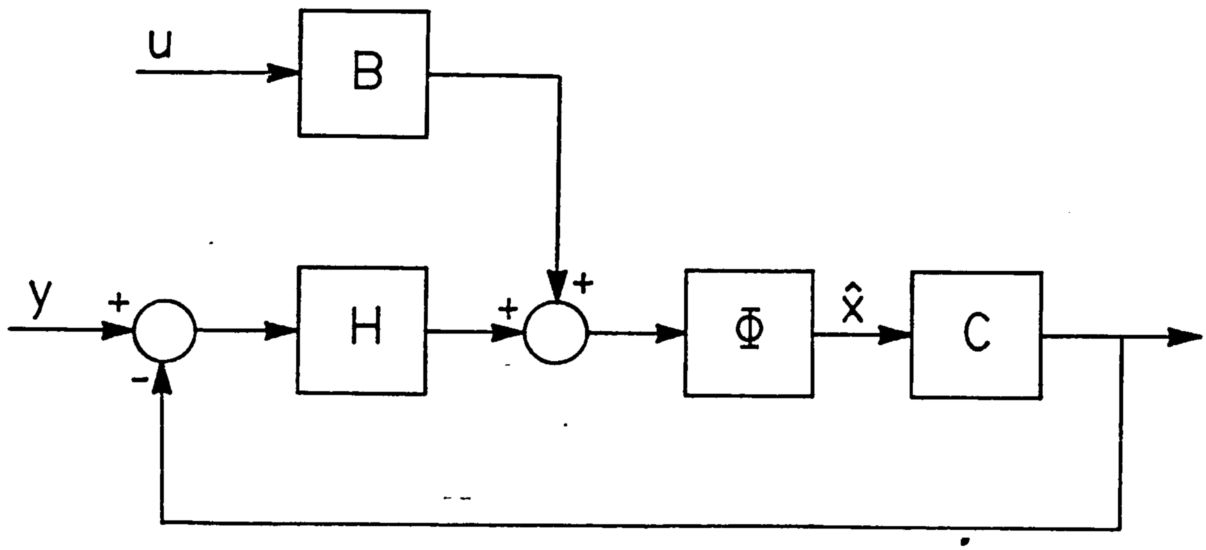


Figure 3-3: Constant Gain Observer Structure

(a) If

$$[A-\nabla f(x)]P + \frac{1}{2} S > 0 \tag{3.94}$$

uniformly for $x \in \mathbb{R}^n$, then the nonlinear observer (estimator) is nondivergent.

(b) If

$$[A-f(\cdot)]P + \frac{1}{2}S > 0 \tag{3.95}$$

uniformly for $x \in \mathbb{R}^n$, then the nonlinear observer is nondivergent.

Theorem 3.14 [43] (Constant Gain Extended Kalman Filter): If uniformly

$$\{\alpha I + [A-\nabla f(x)]\ \} \Sigma + \frac{1}{2} \{\Xi + \Sigma C^T C \Sigma\} > 0, \tag{3.96}$$

then the Constant Gain Extended Kalman Filter (CGEKF) described by

$$0 = \Sigma(A+\alpha I)^T + (A+\alpha I)\Sigma - \Sigma C^T C \Sigma + \Xi, \tag{3.97}$$

$$H = \Sigma C^T \tag{3.98}$$

is nondivergent, with this constant H being used in figure 3-3 as the filter gain.

Remark Theorem 3.14 is basically an approximation result. It says that if a particular observer gain is nondivergent for a particular linear system (A,B,C), and A is close enough to ∇f, then the same H will be a nondivergent filter gain for the nonlinear system. In the case of the CGEKF, the robustness margins are quite substantial, i.e.

not just a "ϵ-close" type of result. The good robustness margins are apparently attributable to the robustness of the Kalman filter, including the time-varying version which we saw in theorem 3.7. We now present a new result on CGEKF robustness margins.

Theorem 3.15 (CGEKF robustness margins) Let the CGEKF satisfy the conditions (3.96-3.98) above, let k be its L_2 nondivergence gain, and suppose there exists a constant $\epsilon>0$ such that

$$|\Sigma^{-1}[\nabla f(x) - A]| \leq \epsilon; \quad \forall\ x\in\mathbb{R}^n. \tag{3.99}$$

Then for all $u_1, u_2 \in \mathcal{L}, \tau\in\mathbb{R}_+$

$$\|[I+C\Phi H]u_1 - [I+C\Phi H]u_2\|_{2,\tau} \geq (1-2\epsilon k)\ \|u_1-u_2\|_{2,\tau}. \tag{3.100}$$

Proof First we examine the equation relating defining the map $u\mapsto x$:

$$\dot{x}(t) = f(x(t)) - HCx(t) + u(t); \quad x(0)=0. \tag{3.101}$$

As we vary u by small amounts, we see that we will be looking at the linearization of (3.101); thus we can apply theorem 2.4 and find the incremental gain (3.100) by looking at the uniform gain of the linear time-varying system:

$$\dot{\xi}(t) = \nabla f(x(t))\xi(t) - HC\xi(t) + Hw(t) \tag{3.102}$$

Define

$$P := \Sigma^{-1} \tag{3.103}$$

$$v(\xi) = \frac{1}{2}\,\xi^T P\xi. \tag{3.104}$$

Then

$$\frac{dv(\xi(t))}{dt} = \dot{v} = \xi^T(t)P\dot{\xi}(t) = \xi^T(t)P[\nabla f(x(t))]\xi(t) + \xi^T(t)Cw, \tag{3.105}$$

and

$$\int_0^T [w^Tw + \xi^T P\Xi P\xi]dt = \int_0^T \left\{ [w+C\xi]^T[w+C\xi] + \xi^T\{-PA-A^TP\}\xi - 2\xi^TCw \right\} d\tau$$

$$= \int_0^T \left\{ [w+C\xi]^T[w+C\xi] + \xi^T\{-PA-A^TP\}\xi - 2\dot{v} + 2\xi^TP[\nabla f]\xi \right\} d\tau$$

$$= \int_0^T \left\{ [w+C\xi]^T[w+C\xi] + 2\xi^T\{P\nabla f-PA\}\xi - 2\dot{v} \right\} d\tau$$

$$\leq \|w+C\xi\|_{2,\tau} - v(\xi(\tau)) + \int_0^T \left\{ 2\xi^TP[\nabla f-A]\Sigma P\xi \right\} d\tau$$

$$\leq \|w+C\xi\|_{2,\tau}^2 + 2k\epsilon\|w\|_{2,\tau}^2 \tag{3.106}$$

and

$$(1-2k\epsilon)\|w\|_{2,\tau} \leq \|[I+C\Phi H]w\|_{2,\tau}. \blacksquare \tag{3.107}$$

Thus we see that if we can bound the deviation of the actual system, ∇f, from a nominal value, A, we can retain some of the benefits of the original Kalman filter robustness. For problems that are not too nonlinear, the CGEKF looks like a good alternative: its nondivergence can be checked (3.96) and it may retain some large robustness margins.

3.3.6 Transformation Based Observers

This section discusses a new type of observer that we will call a transformation based observer. In [26], it is called an observer with "linearizable error dynamics". The basic idea is to transform the nonlinear system (3.6) into a form in which a possible observer is immediately apparent.

Definition A nonlinear system of the form

$$\dot{x}(t) = A\, x(t) + \gamma(y(t),u(t)) \tag{3.108a}$$

$$y(t) = C\, x(t), \tag{3.108b}$$

where $\gamma:\mathbb{R}^m\times\mathbb{R}^m\rightarrow\mathbb{R}^n$, is said to be in *observer form* [51]. Note that the nonlinearity $\gamma(\cdot)$ depends only on the output vector $y(t)$, not the entire state vector.

If we have a system in observer form, it is quite apparent that we can build an observer for it if (A,C) is an observable pair. Consider the observer possibility

$$\dot{\hat{x}}(t) = A\, \hat{x}(t) + \gamma(y(t),u(t)) + H\, [y(t) - C\, \hat{x}(t)] \tag{3.109}$$

The error dynamics for $e=x-\hat{x}$ are given by

$$\dot{e}(t) = [A - HC]\, e(t) \tag{3.110}$$

and so clearly if we select H so that A-HC is stable, we will have exponential decay of the state estimation error.

If we modify the problem slightly to follow our philosophy, we would add some deterministic process and observation noise. Our system is now

$$\dot{x}(t) = A\,x(t) + \gamma(y(t),u(t)) + B\,w(t) \tag{3.111a}$$

$$y(t) = C\,x(t) + d(t). \tag{3.111b}$$

We now can show the following easy result.

Theorem 3.16 The observer (3.109) is a nondivergent observer for (3.111) if we select H so that A-HC is stable.

Proof The error dynamics are

$$\dot{e}(t) = [A - HC]\,e(t) + B\,w(t) + H\,d(t) \tag{3.112}$$

and if A-HC is stable, then (3.112) is L_p-stable for all p. Thus the map $(w,d)\mapsto e$ is stable and the observer is nondivergent. ∎

Remark 1 Unfortunately, most systems that we encounter will not be in observer form. Note, however, that there might exist a state transformation taking our original system (3.6) into observer form (3.109). In [26], conditions are given under which it is possible to transform a given nonlinear system into observer form. Preliminary research has indicated that this conversion is not very generic, i.e., not very many systems can be transformed this way. This is in contradistinction to the dual controller case, to be discussed in section 3.4.5. Even if exact transformation to observer form is not possible, there may be an approximate transformation which could be used with some form of robustness test to guarantee nondivergence.

C-2

Remark 2 If we have the freedom to select our output variables, we can easily put a nonlinear system in observer form. If every state variable is involved in our original nonlinearity $f(\cdot)$, though, this may require the measurement of all the state variables.

We have now discussed various filters with regard to their nondivergence and return difference properties. In section 3.3.2 and appendix D, we covered the optimal nonlinear estimator and showed that while it had good return difference properties, it was far too computationally intensive. Motivated by the relationship between incremental stability and uniform stability of linearized systems as discussed in section 2.3.4, we investigated the extended Kalman filter (EKF) in section 3.3.3 and 3.3.4. We were able to show that the EKF is nondivergent for all M-detectable plants (ones for which a nondivergent model based observer exists) and that furthermore, it possesses some desirable return differnence conditions, making the filter loop $C\Phi H$ a "good" loop operator. In section 3.3.5, we continued by investigating the constant gain model based observers, where we gave conditions for checking the nondivergence of specific observers, involving the checking of the uniform positive definiteness of a matrix for all points in $\mathbb{R}^n$. We then discussed the transformation based observers in section 3.3.6, where the idea is to tranform the plant into a form that allows an observer to be easily built with linear error dynamics. Finally, in appendix C, we discussed some preliminary results on the costate observer, which has some guaranteed loop properties, and looks promising for a nondivergent estimator.

3.4 State Feedback Controllers

3.4.1 Introduction

This chapter has up to now been concerned with nondivergence of estimators. As indicated in section 3.2, that is only half the problem of stabilizing a nonlinear system; the other half is state-feedback. The basic requirement was that a (linear or nonlinear) state-feedback gain be found that made the nonlinear system stable when all the states could be measured. If such a control could be found, we could then use the estimated states instead of the actual states without giving up closed-loop stability. In this section, we will discuss the state feedback problem in more detail. In particular, in sections 3.4.2 and 3.4.3 we discuss an optimal control problem and the guaranteed properties possessed by its solution. We then continue the discussion of state feedback controllers in a dual manner to the estimation material by considering constant linear gain state feedback and transformation based feedback, in sections 3.4.4 and 3.4.5, respectively.

3.4.2 Optimal Control

In this section we will discuss a particular type of state-feedback, that derived from the solution to certain nonlinear optimal control problems.

We first state the nonlinear optimal control problem and its solution [52,53,54].

Optimal Control Problem For the nonlinear system

$$\dot{x}(t) = f(x(t)) + B\,u(t); \quad x(0)=x_0. \tag{3.113}$$

Find u(t) such that the cost functional

$$J(x_0,u(\cdot)) = \frac{1}{2}\int_0^{\infty} [m(x(t)) + \rho u^T(t)u(t)]\,dt \tag{3.114}$$

is minimized with $\rho>0$ and $m(x)\geq 0$ for all x. We denote the minimum such cost J^*:

$$J^*(x_0) \leq J(x_0,u(\cdot)); \quad \forall\, u\in\mathcal{U}. \tag{3.115}$$

Optimal Control Solution If there exists a $V:\mathbb{R}^n\to\mathbb{R}_+$, $V(x)>0$ for all $x\neq 0$, $V(0)=0$ satisfying the *Hamilton-Jacobi-Bellman* (HJB) partial differential equation

$$0 = \frac{1}{2}\,m(x) + V_x^T(x)f(x) - \frac{1}{2\rho}\,V_x^T(x)BB^TV_x(x) \tag{3.116}$$

for all $x\in\mathbb{R}^n$, with $V_x(x)=[\partial V/\partial x]^T$, then the optimal control is

$$u(t) = -\,g(x(t)), \tag{3.117}$$

where g(x) is given by

$$g(x) = \frac{1}{\rho}\,B^TV_x(x). \tag{3.118}$$

and $J^*(x_0)=V(x)$.

Furthermore, if the solution $J^*(x_0)$ exists and is at least twice differentiable, then $J^*(x_0)$ satisfies the HJB equation (3.116) and the optimal control is given by (3.117) with $V(x)=J^*(x)$.

Remark 1 The problem here is that given an initial condition x_0, we are to find the control signal that minimizes (3.114). Thus the problem is formulated such that one is expecting an open-loop optimal time function. It is therefore notable that the solution can be expressed in terms of a feedback function $u=-g(x)$, depending only on the current state, and independent of time.

Remark 2 It is appropriate here to discuss the actual computation of $V(\cdot)$, or $g(\cdot)$ for the optimal control problem. This problem is not nearly as difficult as that of the computation of the optimal nonlinear filtering equations. The optimal filtering equations discussed in section 3.3.2 require the integration of a PDE *in real-time*. This is hard enough to do on a main-frame computer in a laboratory off-line; on an on-line computer in an aircraft, for example, it becomes ridiculous. The HJB equation discussed here is a much easier proposition. Even if we need to use a main-frame computer to solve (3.116), we only have to solve it *once*, off-line, to compute $g(x)$, which can be stored and used in the actual (smaller) on-line computer. Of course, efficiently storing $g(x)$ is a difficult problem in itself, but there are possibilities [55].

While solving the HJB equation is quite difficult to solve with current methods, it is likely that soon there will be better tools [56,57]. As a comparision, the Ricatti equation was considered extremely hard to compute reliably in the early 1960's until the advent of the Schur vector approach [58]. It is hoped that we can provide some motivation here to pursue more efficient methods of calculation and storage of the feedback function $g(x)$.

The motivation that we refer to is in the guaranteed properties possessed by solutions to the optimal control problem presented above. We present these in the next section.

3.4.3 Guaranteed Properties of Optimal Nonlinear Regulators

In this section we present both existing and new results concerning the properties of optimal regulators.

Theorem 3.17 (Guaranteed I/O Properties of Optimal Regulators) Assume that a solution to (3.116) exists. Let G be the nondynamical operator defined by

$$(Gx)(t) := g(x(t)). \tag{3.119}$$

Then the closed-loop system shown in Figure 3-4 has the following properties:

(a) Return Difference Condition [59]

$$\frac{\|[I+G\Phi B]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq 1; \qquad \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+, \ u\neq 0, \tag{3.120}$$

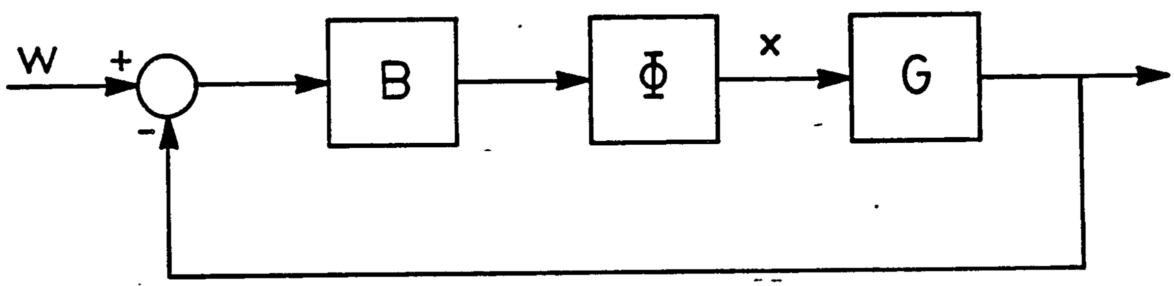


Figure 3-4: The Optimal Control Loop: GΦB

(b) Other Robustness Properties

$$\frac{\|G\Phi B[I+G\Phi B]^{-1}u\|_{2,\tau}}{\|u\|_{2,\tau}} \leq 2; \quad \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+, \ u\neq 0. \tag{3.121}$$

$$\frac{\|[I+(G\Phi B)^{-1}]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq \frac{1}{2}; \quad \forall u\in\mathcal{L}, \tau\in\mathbb{R}_+, \ u\neq 0, \tag{3.122}$$

(c) Robustness Margins

The closed-loop system has -6dB to +∞ multivariable gain margin and -60 to +60 degrees of multivariable phase margin at the plant input.

(d) Closed-loop Stability

$$\|[I+G\Phi B]^{-1}\|_2 \leq 1, \tag{3.123}$$

and

$$\|G\Phi B[I+G\Phi B]^{-1}\|_2 \leq 2, \tag{3.124}$$

i.e. the closed-loop system (mapping $w\mapsto g(x)$) is L_2-stable.

(e) "L_2-domain inequality"

If

$$m(x) = x^T C^T C x, \tag{3.125}$$

then

$$\|[I+G\Phi B]u\|^2_{2,\tau} \geq \|u\|^2_{2,\tau} + \frac{1}{\rho}\|C\Phi Bu\|^2_{2,\tau}; \quad \forall u\in\mathcal{L}, \ \tau\in\mathbb{R}_+. \tag{3.126}$$

(f) Exponential Stability

If there exists an M, γ such that

$$|V_{xx}(x)| \leq M \; ; \quad \forall x \in \mathbb{R}^n, \tag{3.127}$$

$$m(x) + \frac{1}{2\rho} V_x BB^T V_x \;\geq\; \gamma \; |x|^2; \quad \forall x \in \mathbb{R}^n, \tag{3.128}$$

then the closed-loop system (3.113) with (3.117) is exponentially stable and thus for the closed-loop system with $u=-g(x)+v$, the mapping $v \mapsto x$ is L_p-stable for $1 \leq p \leq \infty$.

Proof (a) [59] In a similar manner to theorem 3.7

$$\frac{dV(x(t))}{dt} = V_x{}^T [f(x)+Bu] \tag{3.129}$$

and

$$\int_0^T [u^T u + m(x)] \, dt$$

$$= \int_0^T \left\{ [u+g(x)]^T \rho [u+g(x)] - 2V_x{}^T f(x) - 2V_x^T Bu \right\} dt$$

$$= \int_0^T \left\{ [u+g(x)]^T \rho [u+g(x)] - 2\dot{V} \right\} dt$$

$$= \int_0^T \left\{ [u+g(x)]^T \rho [u+g(x)] \right\} dt - [V(x(\tau))-V(x(0))]. \tag{3.130}$$

Now recall that since we always use a zero initial condition in the definition of the operator Φ, $x(0)=0$, and thus $V(x(0))=0$. We have

$$\|u\|_{2,\tau} \leq \|[I+G\Phi B]u\|_{2,\tau}. \tag{3.131}$$

(b): From (3.131), letting $v=[I+G\Phi B]u$, we have

$$\|[I+G\Phi B]^{-1}v\|_{2,\tau} \leq \|v\|_{2,\tau}. \tag{3.132}$$

Thus

$$\|G\Phi B[I+G\Phi B]^{-1}u\|_{2,\tau} = \| \{I - [I+G\Phi B]^{-1}\} u \|_{2,\tau} \leq 2 \|u\|_{2,\tau} \tag{3.133}$$

and letting $w = G\Phi B[I+G\Phi B]^{-1}u$, we have

$$\|w\|_{2,\tau} \leq 2 \|\{I+(G\Phi B)^{-1}\}w\|. \tag{3.134}$$

(c): The robustness properties are obtained from the return difference condition (a) with the robustness tests in theorem 2.8.

(d): These are immediate from (b) and the definition of norm.

(e): From (3.130), as in (a), $V(x(\tau))\geq 0$, $V(x(0))=0$, and

$$\int_0^{\tau} m(x(t))\, dt = \|C\Phi Bu\|^2_{2,\tau}$$

imply (3.126) after dividing by $\rho>0$.

(f): The conditions (3.127-3.128) guarantee that $V(x)$ is a Lyapunov function for the closed-loop system satisying the requirements of theorem 2.2 and thus $x=0$ is an exponentially stable equilibrium point of the closed-loop system. We can now conclude L_p-stability by theorem 2.3. ∎

These theorems give us the guaranteed properties that are possessed by the optimal control solution. They are expressed in terms of the loop operator $G\Phi B$. The proof of result (a) was taken from [59]. The result (c) was also found by [60] for a more general nonlinear system in a Lyapunov sense only. The gain margins of (c) were also found for the Lyapunov case by [61,62]. Note that here we have more than just gain and phase margins, we have a "ball" of robustness, defined by (a) and (b) and by the robustness tests of section 2.5, and in addition, we have I/O stability, not Lyapunov stability.

Result (e), the "L_2 domain inequality" is the nonlinear extension of the Kalman frequency domain inequality [2,72] for linear systems. This inequality (3.126) may be the start of loop shaping ideas, as it relates properties of the solution to the HJB equation ($G\Phi B$) to properties of the parameters of the problem statement ($C\Phi B, \rho$).

In addition to the results of theorem 3.17, we would like to have incremental versions, i.e., prove that the closed-loop system is incrementally stable. Unfortunately, these results are not available at this time. It seems that it will require some sort of one-to-one mapping condition on the nonlinearity f. Note that for the one-dimensional case (n=1), the HJB can be solved explicitly, and the solution is incrementally stable if and only if ∇f does not change sign over the real line. This incremental stability issue is related to the results of appendix C concerning the costate observer.

The property (f) is stated in a global manner, but if the conditions (3.127-3.128) hold only in some ball (if V is smooth then the condition (3.127) holds in any bounded region) then we will get a small-signal form of I/O stability, as stated in theorem 2.3. This will be the case for systems containing saturations, as we know that the system can never be stable when signals can be arbitrary in size and injected into any component of $\dot{x}$.

Now consider the optimal cost $J^*(x)=V(x)$ as a function of the control weight, ρ. Kwakernaak and Sivan [63] discuss the limiting behavior of V(x) as $\rho \to 0$ for the linear case, where $V(x)=(1/2)x^TKx$ and K is the solution to the appropriate Ricatti equation for the Linear-Quadratic-Regulator (LQR) problem. Since the HJB equation is the generalization of the Ricatti equation for nonlinear systems, we might suspect that there would be some similar asymptotic behavior here as well. There is:

Theorem 3.18 (Cheap Control) Let

$$m(x) = \frac{1}{2} x^T C^T Cx \tag{3.135}$$

and consider the value of the optimal cost $J^*(x)=V^\rho(x)$ as we vary the control weight, ρ, down to zero. Then the following facts hold:

(a) The following limit always exists

$$\lim_{\rho\to 0} V^{\rho}(x) = V^{0}(x). \tag{3.136}$$

(b) If

$$\lim_{\rho\to 0} V_x^{\rho}(x) = 0, \tag{3.137}$$

then there exists a unitary matrix $W_{\rho}(x)$, depending on both x and ρ (i.e. $W_{\rho}^{T}(x)W_{\rho}(x) = I \;\forall x,\rho$) such that

$$\lim_{\rho\to 0} \sqrt{\rho}\; W_{\rho}(x)g_{\rho}(x) = Cx; \quad \forall\; x\in\mathbb{R}^n. \tag{3.138}$$

Proof (a): Fix $x\in\mathbb{R}^n$. Then $V^{\rho}(x)$ is non-increasing with decreasing ρ by inspection of (3.114). If we were to use the same feedback $g(\cdot)$ for a smaller value of ρ, we would get a smaller cost because the trajectories would be the same and ρ is the control weight. Since $V^{\rho}(x)$ is the optimal (minimum) cost starting from x,

$$\rho_1 < \rho_2 \quad \Longrightarrow \quad V^{\rho_1}(x) \le V^{\rho_2}(x); \quad \forall x\in\mathbb{R}^n. \tag{3.139}$$

Since $V^{\rho}(x) \ge 0 \;\forall x$, the limit must exist.

(b): Letting $V_x^{\rho}(x) \to 0$ in the HJB equation (3.116), we have

$$\lim_{\rho\to 0} \left\{ B^T \frac{V_x^{\rho}}{\sqrt{\rho}} \right\}^T \left\{ B^T \frac{V_x^{\rho}}{\sqrt{\rho}} \right\} = \lim_{\rho\to 0} |\ell^{\rho}(x)|^2 = x^T C^T C x \tag{3.140}$$

where ℓ is defined in the obvious way. Since this implies that

$$|\ell^{\rho}(x)| = |Cx|; \quad \forall x, \rho, \tag{3.141}$$

there must exist a unitary matrix $W_{\rho}(x)$ depending on x and ρ so that

$$\lim_{\rho \to 0} \sqrt{\rho}\, W_{\rho}(x) g_{\rho}(x) = Cx; \quad \forall x. \quad \blacksquare \tag{3.142}$$

Remark 1 We would like to have $V^0(x)=0$ imply (3.137) but need some uniformity condition for the convergence. It seems likely to occur in practice, as we would not expect $V^{\rho}(x)$ to become too "oscillatory" in x as $\rho \to 0$.

Remark 2 In the linear case [63], the above results are related to the minimum phaseness of the original open-loop system, namely, if $C\Phi B$ is minimum phase, then $V^{\rho} \to 0$ as $\rho \to 0$ and (3.137) holds. We discuss this issue further in section 4.5

3.4.4 Constant Linear State Feedback

In this section we present results on the stability of a state feedback controller with a constant linear gain. The results are completely dual to those of section 3.3.5.

Theorem 3.19 (Linear-Quadratic-State-Feedback [6]) Let the linear gain matrix G be defined by

$$G = R^{-1}B^TK \tag{3.143}$$

where $K=K^T \geq 0$ is the solution to the Ricatti equation

$$0 = K(A+\alpha I) + (A+\alpha I)^TK + Q - KBR^{-1}B^TK \tag{3.144}$$

with $\alpha>0$, $Q=Q^T \geq 0$, and $R=R^T>0$. Then if uniformly

$$K[\alpha I + (A - \nabla f(x)) + \frac{1}{2} [Q + KBR^{-1}B^TK] > 0 \tag{3.145}$$

for all $x \in \mathbb{R}^n$, then the feedback

$$u(t) = - G\, x(t) \tag{3.146}$$

stabilizes the nonlinear system

$$\dot{x}(t) = f(x(t)) + B\, u(t) + w(t) \tag{3.147}$$

$w \mapsto x$, i.e. there exists k such that for the closed-loop system

$$\|x\|_\tau \leq k\, \|w\|_\tau; \quad \forall w \in \mathcal{L}, \tau \in \mathbb{R}_+. \tag{3.148}$$

Proof See [6]. ∎

Note that we also could have a robustness margin result akin to theorem 3.15, which we will not bother to state here as the analogy should be quite clear. Thus it appears that under some conditions, including systems that are not too nonlinear, we can design a state feedback controller quite simply.

3.4.5 Transformation Based State Feedback

Continuing the analogy with the treatment of observers, we now discuss a state feedback that is analogous to that discussed in section 3.3.6.

Definition [51] A nonlinear system of the form

$$\dot{x}(t) = A\, x(t) + B\, \alpha(x(t)) + B\, u(t) \tag{3.149}$$

is said to be in *controller form*.

We can stabilize a system of this type quite easily, because we can cancel the nonlinear term $\alpha(x)$ through the control u.

Theorem 3.20 The state feedback

$$u(t) = -\,G\, x(t) - \alpha(x(t)) \tag{3.150}$$

will stabilize the system in controller form

$$\dot{x}(t) = A\, x(t) + B\, \alpha(x(t)) + B\, u(t) + B\, w(t) \tag{3.151}$$

(i.e. (3.148) holds), if A-BG is stable.

Proof The closed loop trajectory is

$$\dot{x}(t) = [\, A - BG \,]\, x(t) + B\, w(t). \quad \blacksquare \tag{3.152}$$

Remark Note that many mechanical systems, such as robots, are already in controller form, where all the nonlinearities are in the acceleration equation. Even if our system is not originally in controller form, there may exist a state transformation that takes the system into controller form. Just as in the observer case, there exist conditions that make it possible to check whether it is possible to transform a given system into controller form. In [21,22,23] conditions are given under which a combination of state transformation and nonlinear state feedback will linearize a system. Just the state transformation part is what we need to get to controller form—the state feedback part corresponds to the function $\alpha(\cdot)$ above.

In this chapter we have covered both state estimation, and state feedback. We can now put them together to produce a stable closed-loop system, as indicated by the separation results of section 3.2. However, we have no way of telling, a priori, what the good properties of the closed-loop system will be, if any. The next chapter will explain how to get these a priori guarantees in a systematic manner, in which we try to exploit the good properties possessed by either the estimator loop or the state feedback loop discussed in this chapter.

CHAPTER 4. THE NMBC/LOR METHODOLOGY

4.1 Introduction

So far, we have covered both the analysis of feedback systems in chapter 2 and the synthesis of stabilizing compensators in chapter 3. In this chapter we attempt to bring the analysis results to bear on the guaranteed properties we uncovered in the last chapter. We now take as our goal the design of a dynamic nonlinear compensator that will meet given specifications for the closed-loop system.

The structure of this chapter is as follows. We start off in section 4.2 by presenting the technical theorems of Loop Operator Recovery (LOR) for the three variations that we will discuss: (1) recovery at the plant input, (2) recovery at the plant output, and (3) formal loop shaping. In addition we present a result on Q-parameterization [28] that will be useful. After presenting the basic theorems, we will be ready to detail the entire design process using the NMBC/LOR methodology, in section 4.3. Section 4.4 gives some design hints, involving scaling and dynamic augmentation, while section 4.5 gives an informal discussion of "minimum phase" systems. Finally, section 4.6 critiques the entire NMBC/LOR procedure and compares it to some other possibilities.

4.2 The Loop Operator Recovery Theorems

The Loop Operator Recovery (LOR) theorems that will be presented here are direct extensions of the Loop Transfer Recovery procedure of LQG/LTR [1,2] in the linear case. The driving force behind the LOR idea thus comes directly from [1], although the methods of proof here are of necessity different than their counterparts in the linear case. A study of the methods of proof used here might be instructive for readers interested in linear systems as the proofs here are different from the original ones, even when specialized to the linear case.

For the following, we assume that we have a plant

$$\dot{x}(t) = f(x) + B\,u(t) + B\,w(t) \tag{4.1a}$$

$$y(t) = C\,x(t) + d(t), \tag{4.1b}$$

and the nonlinear model-based-compensator

$$\dot{z}(t) = f(z(t)) - HCz(t) - Bg(z(t)) + H[y(t)-r(t)] \tag{4.2a}$$

$$u(t) = -\,g(z(t)), \tag{4.2b}$$

or, in operator notation

$$y = C\Phi B(u+w) + d \tag{4.3}$$

$$u = -\,G[\Phi^{-1} + HC + BG]^{-1}H\,(y-r), \tag{4.4}$$

or

$$y = P(u+w) + d \tag{4.5}$$

$$u = K\,(r-y) = K\,e, \tag{4.6}$$

where

$$P := C\Phi B \tag{4.7}$$

$$K := -\,G[\Phi^{-1} + HC + BG]^{-1}H\ (-I), \tag{4.8}$$

and where we define the nonlinear (nondynamical) operator G:

$$(Gx)(t) := g(x(t)). \tag{4.9}$$

See figure 4-1 for a block diagram of the closed-loop system. Note that we have not used $\hat{x}$ as the state of the compensator. This is because we are using a reference command, and thus the state of the compensator is no longer the state estimate. However, this does not bother us, as we are interested in closed-loop performance, etc., not state estimation.

We now present a basic operator fact before presenting the LQR theorems.

Fact For Φ,G nonlinear and B linear,

$$[\Phi^{-1}+BG]^{-1}B = \Phi B[I+G\Phi B]^{-1}. \tag{4.10}$$

Proof The proof of this fact is quite easy, but it will be instructive to do it by inspection of the block diagram. Consider figure 4-2. We have

$$x = \Phi B[w-Gx] \tag{4.11}$$

$$\Phi^{-1}x = Bw - BGx \tag{4.12}$$

$$x = [\Phi^{-1}+BG]^{-1}Bw. \tag{4.13}$$

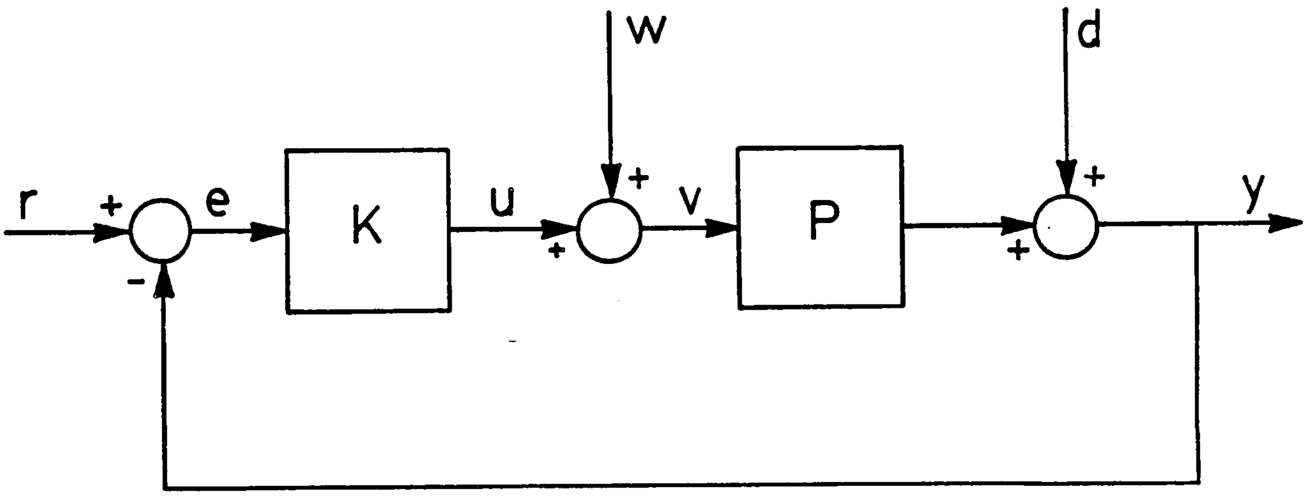


Figure 4-1: Closed-Loop System

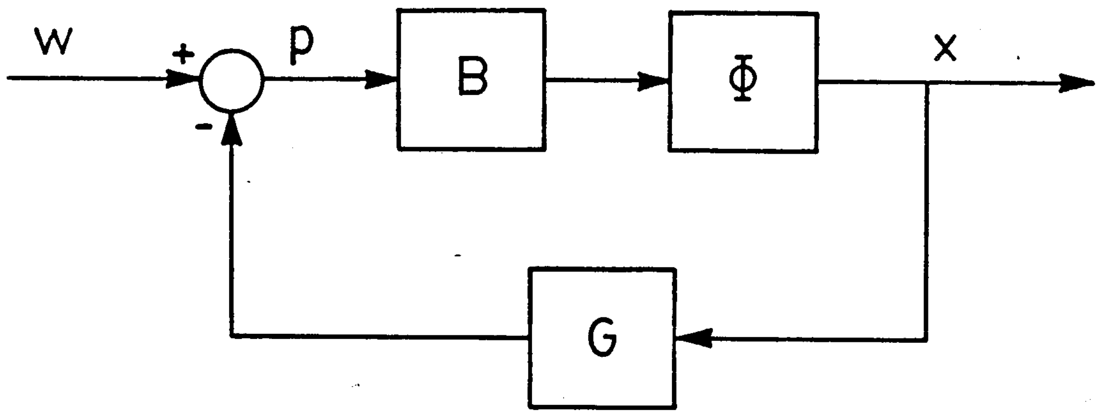


Figure 4-2: Demonstration of Operator Equality

Similarly,

$$p = w - G\Phi B\ p \tag{4.14}$$

$$p = [I+G\Phi B]^{-1}\ w \tag{4.15}$$

$$x = \Phi B \cdot p = \Phi B[I+G\Phi B]^{-1}\ w, \tag{4.16}$$

and thus we have established the fact. ∎

Consider figure 4-3, where we depict the NMBC structure in the closed-loop system. We now present two theorems dealing with the asymptotic behavior of the loop operators for figure 4-3.

Theorem 4.1 (LOR at the Plant Input) Let H_μ, the filter gain in the compensator, be a linear operator, parameterized by μ such that

$$\lim_{\mu\to 0} H_\mu\sqrt{\mu} = BW, \tag{4.17}$$

where W is any invertible operator. Then if B is linear

$$z \to x \quad \text{as} \quad \mu\to 0, \text{ if } d,r=0, \tag{4.18a}$$

and

$$\lim_{\mu\to 0} (-K)(-P) = G\Phi B. \tag{4.18b}$$

For this theorem, we do not require that C be linear.

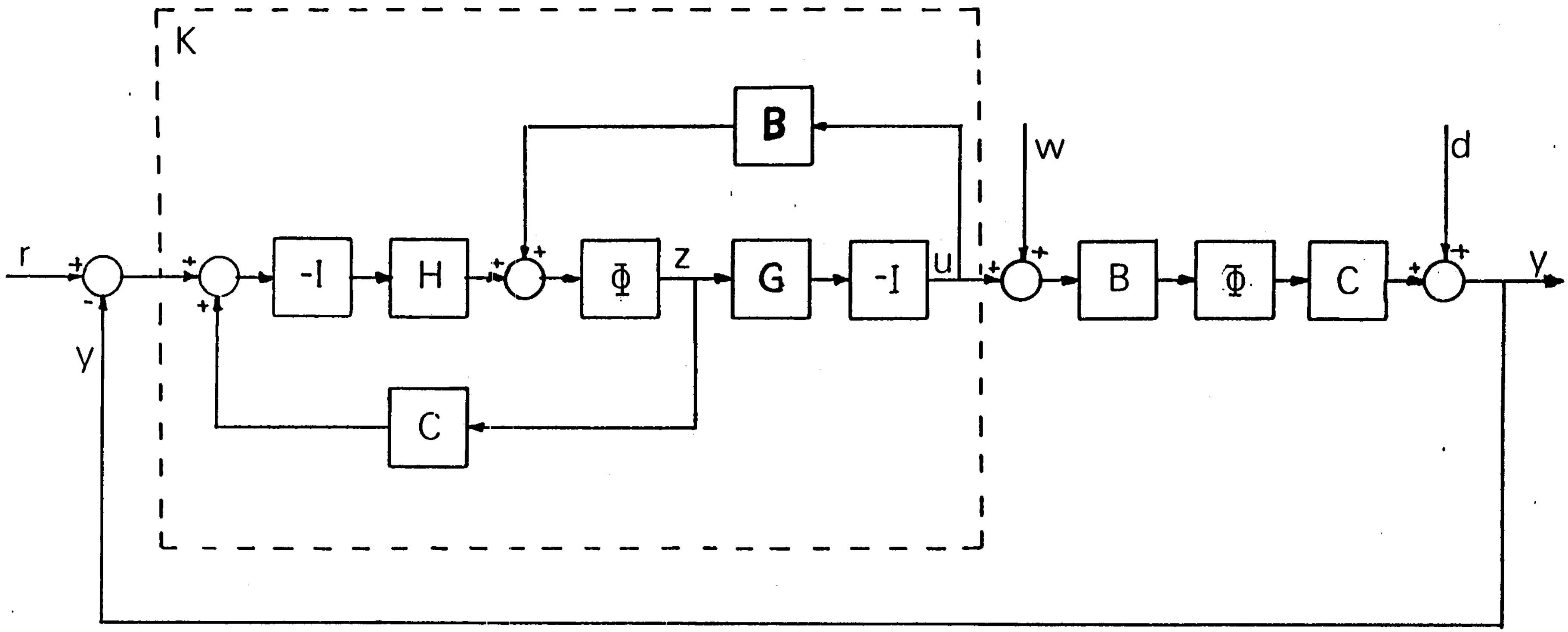


Figure 4-3: The NMBC Structure

Proof Define the nonlinear operator

$$X := [\Phi^{-1} + BG]^{-1}. \tag{4.19}$$

Then

$$K = - G[X^{-1} + HC]^{-1}H\,(-I) \tag{4.20}$$

and by (4.10) and the linearity of H,

$$\begin{aligned} K &= - GXH[I+CXH]^{-1}(-I) \\ &= - GXH\sqrt{\mu}[I\sqrt{\mu} + CXH\sqrt{\mu}]^{-1}(-I). \end{aligned} \tag{4.21}$$

Now let $\mu \to 0$, and we get

$$\begin{aligned} K &\to - GXBW[CXBW]^{-1}(-I) = - GXB[CXB]^{-1}(-I) \\ &= - G[\Phi^{-1}+BG]^{-1}B\cdot\left\{ C[\Phi^{-1}+BG]^{-1}B \right\}^{-1}(-I) \\ &= - G\Phi B[I+G\Phi B]^{-1}\cdot\left\{ C\Phi B[I+G\Phi B]^{-1} \right\}^{-1}(-I) \\ &= - G\Phi B\{C\Phi B\}^{-1}(-I). \end{aligned} \tag{4.22}$$

where we used (4.10) again. Thus

$$(-K)(-P) \to G\Phi B\{C\Phi B\}^{-1}\{C\Phi B\} = G\Phi B, \tag{4.23}$$

where the convergence is pointwise in the signal space and time, i.e.

$$\lim_{\mu \to 0} \left\| \left\{(-K)(-P) - G\Phi B \right\} u \right\|_{p,\tau} = 0 \tag{4.24}$$

for each fixed $u \in \mathcal{L}, \tau \in \mathbb{R}_+$. (4.18a) is shown by repeating the above for G=I. ∎

Remark The convergence is only pointwise, not uniform. Thus it is only a formal result if we do not have any guarantee of closed-loop stability. Note also that the result holds independent of the gain G. That is, G can be nonlinear, time-varying, or even a dynamical system itself.

Theorem 4.2 (LQR at the plant output) Let G_ρ be parameterized by $\rho>0$. Let assumptions (a-e) be

(a) $\lim_{\rho\to 0} \sqrt{\rho}\, G = W_1 C$; W_1 invertible,

(b) $\lim_{\rho\to 0} G\sqrt{\rho} = W_2 C$; W_2 invertible,

(c) B linear,

(d) C linear,

(d) $[\Phi^{-1}+BG_\rho]^{-1}$ linear $\forall\ \rho>0$,

(e) $G_\rho[x_1-x_2] \to G_\rho x_1 - G_\rho x_2$ if $C(x_1-x_2) \to 0$.

Then

(i) (a) and (c) imply that

$$\lim_{\rho\to 0} Cz_\rho = 0, \text{ if } w=0. \tag{4.25}$$

(ii) (a-e) imply

$$\lim_{\rho\to 0} PK = C\Phi H. \tag{4.26}$$

Proof First we show conclusion (i). Since we will consider we will be considering the loop PK, we are implicitly assuming w=0.

$$x_\rho = \Phi B(-Gz_\rho) = \Phi B(-Gx_\rho+Gx_\rho-Gz_\rho) \tag{4.27}$$
$$\Phi^{-1}x_\rho = -BGx_\rho + B(Gx_\rho-Gz_\rho) \tag{4.28}$$
$$x_\rho = [\Phi^{-1}+BG]B[Gx_\rho-Gz_\rho] = \Phi B[I+G\Phi B]^{-1}[Gx_\rho-Gz_\rho] \tag{4.29}$$

and

$$Cx_\rho = C\Phi B[I\sqrt{\rho} + \sqrt{\rho}G\Phi B]^{-1}[\sqrt{\rho}Gx_\rho-\sqrt{\rho}Gz_\rho]. \tag{4.30}$$

Now, let $\rho\to0$, and

$$Cx_\rho \to C(x_\rho-z_\rho). \tag{4.31}$$

Therefore, Cz_ρ must go to zero. Consider implication (ii).

Lemma $G[\Phi^{-1}+BG]^{-1} \to (C\Phi B)^{-1}C\Phi$ as $\rho\to0$. (4.32)

Proof of Lemma We have

$$G[\Phi^{-1}+BG]^{-1} = G[\Phi^{-1}+BG]^{-1}[\Phi^{-1}+BG-BG]\Phi \tag{4.33}$$
$$= G\{I - [\Phi^{-1}+BG]^{-1}BG\}\Phi, \tag{4.34}$$

by linearity of $[\Phi^{-1}+BG]^{-1}$. Now, consider

$$C\{I - [\Phi^{-1}+BG]BG\}\Phi = \{C - C\Phi B[I+G\Phi B]^{-1}G\}\Phi$$
$$= \{C - C\Phi B[I\sqrt{\rho} + \sqrt{\rho}G\Phi B]^{-1}\sqrt{\rho}G\}\Phi$$
$$\to \{C - C\}\Phi = 0, \text{ as } \rho \to 0. \qquad (4.35)$$

Thus we can use condition (e) to conclude

$$G(-I)[\Phi^{-1}+BG]^{-1} \to - \{G - G[\Phi^{-1}+BG]^{-1}BG\}\Phi$$
$$= - \{G - G\Phi B[I+G\Phi B]^{-1}G\}\Phi$$
$$= - \{I - G\Phi B[I+G\Phi B]^{-1}\}G\Phi$$
$$= - [I+G\Phi B]^{-1}G\Phi$$
$$\to - (C\Phi B)^{-1}C\Phi, \text{ as } \rho \to 0. \ \blacksquare \text{ (lemma)} \qquad (4.36)$$

We now put these results together:

$$PKe = C\Phi B\{-G[\Phi^{-1}+HC+BG]^{-1}H(-I)\}e = C\Phi B\{-G(-I)[\Phi^{-1}+BG][- HCz + He]$$
$$\to C\Phi B\{-G(-I)[\Phi^{-1}+BG]^{-1}He$$
$$\to C\Phi B(-I)(-I)(C\Phi B)^{-1}C\Phi H = C\Phi H. \ \blacksquare \qquad (4.37)$$

Remark This theorem appears somewhat limited in scope because of the conditions (d) and (e). These imply that our system is of the form

$$\dot{x}(t) = A\, x(t) + B\alpha(Cx(t)) + B\, u(t), \qquad (4.38)$$

which is both in controller and observer form (see sections 3.3.6 and 3.4.5). If our system is not in this form, we may get convergence only over some limited set of signals.

We now come to the Formal Loop Shaping (FLS) theorem. Let the system $T_{des}: v \mapsto w$ be given by

$$\dot{s}(t) = f_1(s(t)) + H\, v(t) \tag{4.39a}$$

$$w = Ds(t). \tag{4.39b}$$

Let Ψ be the Φ operator for the system (4.39a), i.e.

$$\Psi := [S^{-1}+F_1]^{-1}, \tag{4.40}$$

where S is the integral operator and $(F_1x)(t)=f_1(x(t))$. Then

$$w = T_{des}\, v = D\Psi H\, v. \tag{4.41}$$

We will be considering the block diagram shown in figure 4-4 for our control system under FLS. The equations are

$$u = [I+G_x\Phi B]^{-1}G_s\Psi H[I+D\Psi H]^{-1}[e+C\Phi Bu] \tag{4.42}$$

and thus

$$K = \left\{ [I+D\Psi H][G_s\Psi H]^{-1}[I+G\Phi_x B] - C\Phi B \right\}^{-1}. \tag{4.43}$$

Comparing figures 4-3 and 4-4, we can see that if everything was linear, and $C\Phi H=D\Psi H$, the NMBC/LQR and FLS compensators would be equivalent. That is, FLS for the linear case is standard LQG/LTR if

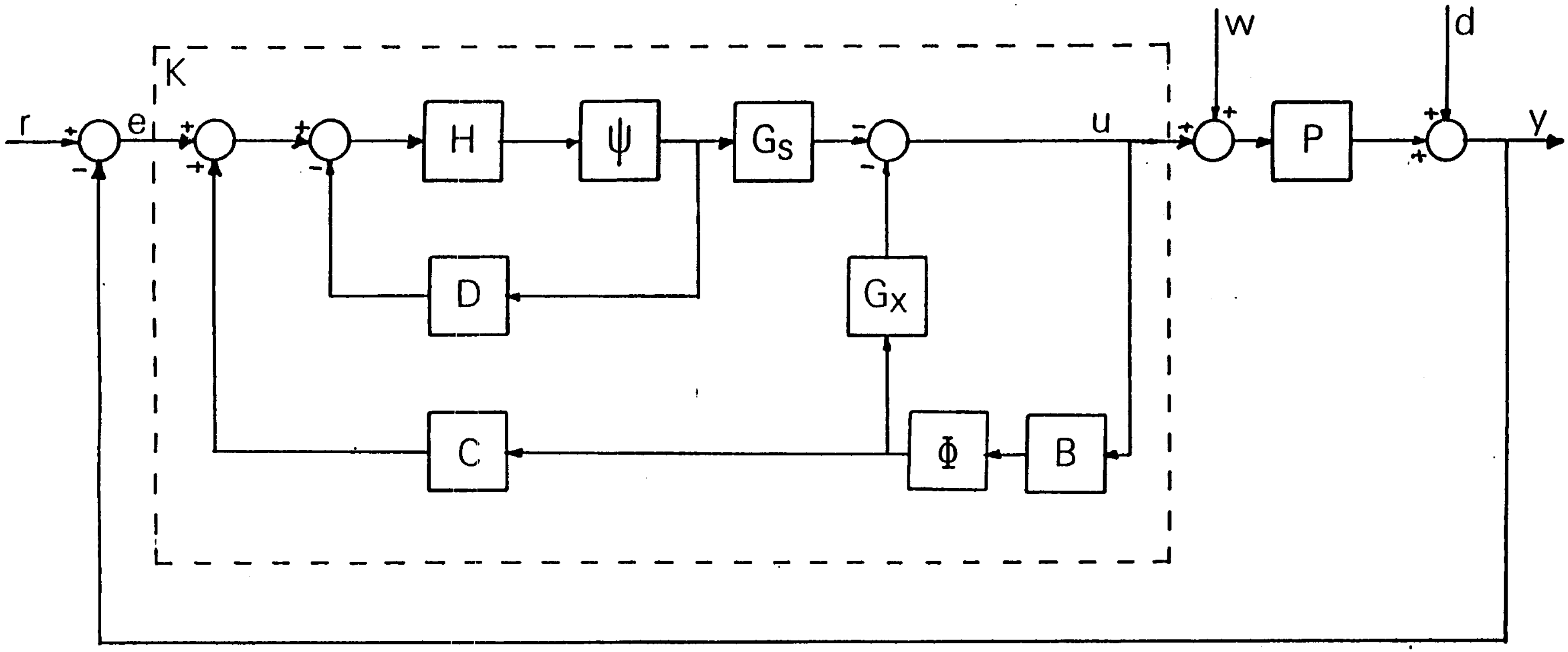


Figure 4-4: Formal Loop Shaping Structure

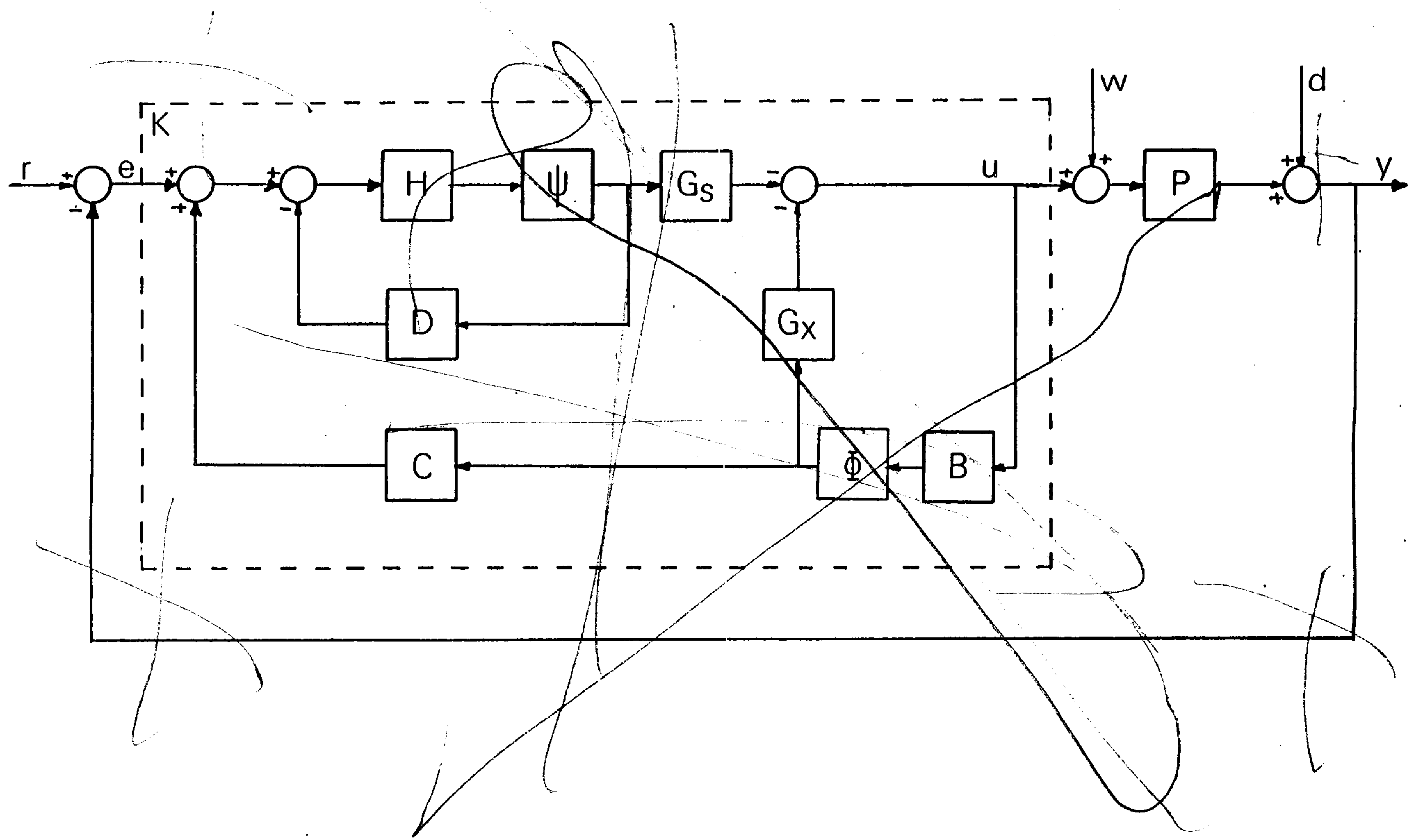


Figure 4-4: Formal Loop Shaping Structure

we let $C\Phi H = D\Psi H$. Since we are dealing with nonlinear systems here, we expect that figure 4-3 and 4-4 have different properties, even if $C\Phi H = D\Psi H$. They do.

Theorem 4.3 (Formal Loop Shaping) Consider the control system block diagram shown in figure 4-4, where K is given by (4.43). Then if there is an invertible operator W such that

$$\text{(a)} \quad \lim_{\rho \to 0} \sqrt{\rho} G_x = WC, \quad \text{and} \tag{4.44}$$

$$\text{(b)} \quad \lim_{\rho \to 0} \sqrt{\rho} G_s = WD, \tag{4.45}$$

then

$$\lim_{\rho \to 0} PK = D\Psi H. \tag{4.46}$$

Proof

$$K = (C\Phi B)^{-1} \left\{ [I+D\Psi H][G_s \Psi H]^{-1}[I+G\Phi_x B](C\Phi B)^{-1} - I \right\}^{-1}, \tag{4.47}$$

so that

$$\begin{aligned} PK &= \left\{ [I+D\Psi H][\sqrt{\rho}G_s \Psi H]^{-1}[\sqrt{\rho}I+\sqrt{\rho}G_x \Phi B](C\Phi B)^{-1} - I \right\}^{-1} \\ &\to \left\{ [(D\Psi H)^{-1}+I](C\Phi B)(C\Phi B)^{-1} - I \right\}^{-1} \\ &= D\Psi H. \quad \blacksquare \end{aligned} \tag{4.48}$$

We have now presented the technical theorems for the recovery procedure. We now present a result from Q-parameterization [28], which will allow us to guarantee that FLS produces a closed-loop stable system.

Theorem 4.4 (Q-Parameterization) Consider the closed-loop system of figure 4-5. If Q is stable and P is incrementally stable, then the system will be closed-loop stable.

Proof We have

$$
\begin{aligned}
\|u\|_T &= \|Q\,[r + P\,u - d - P(u+w)]\|_T \\
&\leq \|Q\|\cdot[\ \|r\|_T + \|d\|_T + \|P\|_\Delta\cdot\|w\|_T\ ] \\
&\leq k_1\|(r,d,w)\|_T,
\end{aligned}
\tag{4.49}
$$

and

$$
\begin{aligned}
\|y\|_T &= \|d + P(u+w)\|_T \leq \|d\|_T + \|P\|\cdot\|w\|_T + \|P\|\cdot\|u\|_T, \\
&\leq k_2\|(r,d,w)\|_T. \quad \blacksquare
\end{aligned}
\tag{4.50}
$$

In the next section we will discuss the use of these technical results, together with the results of the previous chapters, in designing feedback control systems.

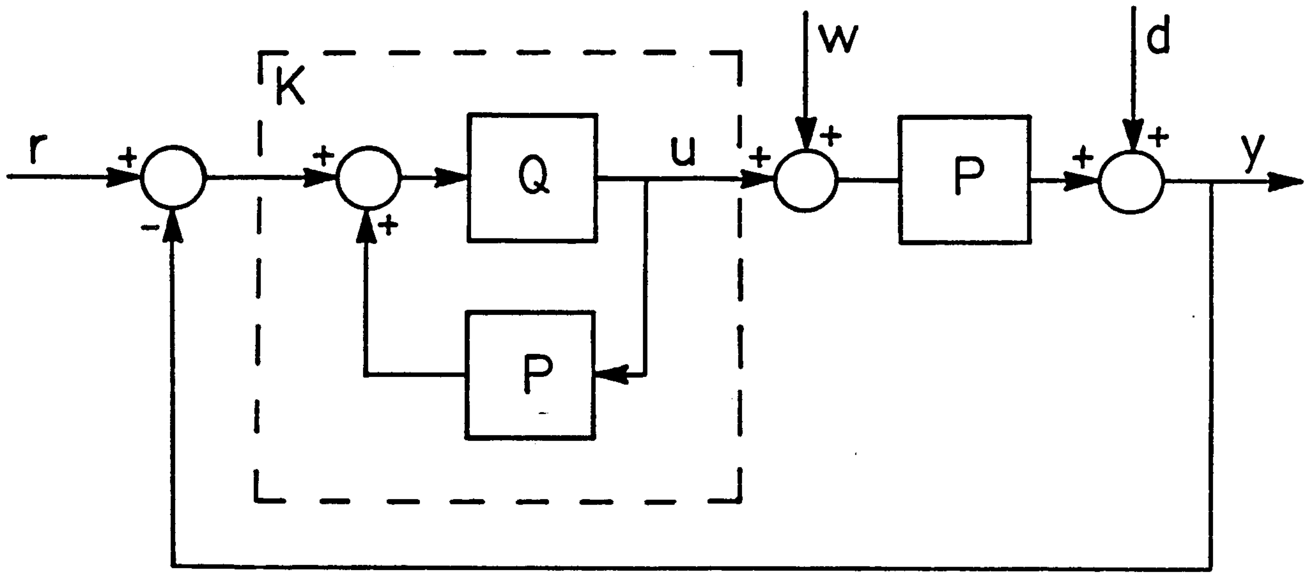


Figure 4-5: Q-Parameterization Structure

4.3 The NMBC/LQR Methodology

4.3.1 Introduction

In this section we will assemble all of our results together to indicate the step-by-step procedure for designing a multivariable feedback control system. We have three variations. The first one, recovery at the plant input, is discussed in section 4.3.2. This is the most general procedure of the three, in that it does not require the special conditions that we saw in theorem 4.2, nor does require a stable plant. The second procedure, recovery at the plant output, is potentially more useful in that it allows the command following and output disturbance properties to be manipulated, although it does have the restrictions of theorem 4.2. This procedure is described in section 4.3.3. Finally, the formal loop shaping procedure, discussed in section 4.3.4, allows the command following and output disturbance rejection loop (the loop broken at the plant output) to be shaped arbitrarily, but has the restriction of requiring an open-loop incrementally stable plant to guarantee global closed-loop stability.

4.3.2 Recovery at the Plant Input

We now give the steps for designing a controller based on the recovery at the plant input method, using theorem 4.1.

Step 1 (Modeling): Develop a Model for the plant to be controlled as a nonlinear differential equation, or equivalently, in the form $P=C\Phi B$, discussed in section 2.2. This is our design plant model, and can include augmented dynamics (which will be discussed in section

4.4). As part of the modeling process we must develop bounds on our modeling error. This is most easily done (for now) in the frequency domain.

Step 2 (Specifications): Convert all available specifications into specifications on the loop broken at the plant input, or T=(-K)(-P). Consult figure 4-1. We will call this the *desired loop* T_{des}. Obviously, if the more easily we can express our specifications in terms of T_{des}, the better off we will be. The specifications that can be easily translated in terms of T_{des} are

(a) Nominal Closed-Loop Stability. Obviously T_{des} must be closed-loop stable. In addition, since it will be realized as (-K)(-P) for some K, that realization must be internally closed-loop stable as well.

(b) Input disturbance specifications. If T_{des} is large for all signals in some set, then the response to noises in that set will be small. Use the results of section 2.4 for calculations.

(c) Robustness to unmodeled dynamics. We develop specifications for bandwidth, etc. and can express them in terms of T_{des} using the results of section 2.5.

The basic idea of this step is to capture *all* available specifications. Thus, if a loop T_{des} meets these specifications, we will be satisfied using it.

Step 3 (Target Loop Design): In this step we delovelop a target loop that will meet all of the specifications of step 2. In this variation of NMBC/LOR, we use a loop

$$T_{des} = G\Phi B \tag{4.51}$$

for some nonlinear operator G. It is suggested that this G come from the solution to an optimal regulator problem, although this is by no means necessary. However, if we did use optimal regulator theory, as discussed in section 3.4, we would be guaranteed several good properties for the target loop, including

(a) Nominal Closed-Loop Stability. The regulator loop is guaranteed to be closed-loop stable.

(b) Adjustable Performance. By varying the matrix Q and gain ρ in the optimal regulator problem, we can adjust the performance of the target loop T_{des}. For example, by making ρ smaller, we increase the performance and bandwidth. In the linear case, bandwidth control is many times sufficient to develop a reasonable "first pass" controller.

(c) Guaranteed Robustness. As shown in theorem 3.17, the loop $G\Phi B$ has many guaranteed properties, including $(1/2,\infty)$ gain margin and +/-60 degrees phase margin. Thus we automatically have obtained a robust target loop.

Step 4 (NMBC Construction): We now construct the NMBC as shown in figure 4-3. The gain G comes from step 3, and the gain H must be chosen such that the closed-loop system is stable. The way to do this is to use the separation theorem of section 3.2 to realize that we should select H to make the estimator contained in the NMBC nondivergent. This could be done in any of the ways discussed in section 3.3, however, the easiest general purpose idea is to use the extended Kalman filter. From theorem 3.6, we know that if the system is detectable, then the EKF will be nondivergent. The CGEKF, while computationally simpler, may not be nondivergent: we need to check the conditions given in section 3.3.5. As required for the next step we will need a set of H's, parameterized by $\mu>0$, such that

$$\lim_{\mu\to 0} \sqrt{\mu}\, H_{\mu} = BW, \quad W \text{ invertible}, \tag{4.52}$$

and H_{μ} produces a nondivergent estimator for all $\mu>0$. These H_{μ}'s can be generated by the EKF if we select $\Xi=qBB^T$, with $q=1/\mu$, and we meet certain minimum phase conditions (see section 4.5) analogous to the minimum phase conditions in time-invariant linear systems.

Step 5 (Loop Operator Recovery): In this step, we apply theorem 4.1 and let μ get small. We have that

$$\lim_{\mu\to 0} (-K)(-P) = G\Phi B = T_{des}. \tag{4.53}$$

If we select μ sufficiently small, then our actual loop, $(-K)(-P)$, will approach the target loop, T_{des}, that we selected for its good characteristics. Furthermore, we know that the system will be closed-loop stable (including internally stable) for all values of $\mu>0$. Thus, for our selected value of μ, we have designed a compensator K that stabilizes the plant P and meets all of the specifications.

Step 6 (System Simulation): In order to be sure that all specifications were in fact included in step 2 and that nothing was neglected, many simulations of the closed-loop system must be performed. This will involve more actual simulations than in the linear case, because it will be harder to "excercise" all the different modes and operating regimes. If the design proves unsatisfactory in any manner, we must go back to step 2 or step 3 to modify the target loop $G\Phi B$ by modifying G. Because $G\Phi B$ has half the dimension of (-K)(-P), it is easier to simulate and check the specifications on $G\Phi B$ in step 3, rather than waiting until step 6.

In section 5.3, we present a numerical example demonstrating this recovery procedure using a simple nonlinear pendulum model.

End of Design Procedure

We have outlined the design procedure for loop operator recovery at the plant input, which is used when specifications can most easily be posed on the loop (-K)(-P).

4.3.3 Recovery at the Plant Output

We now give the steps for designing a controller based on the recovery at the plant output method, using theorem 4.2. We will parallel the previous section as much as possible and will leave out material that would be repetitive. The recovery at the plant output method is to be used if the plant model is in *both* controller and observer form so that theorem 4.2 applies, and if the specifications are most naturally posed on the loop broken at the plant output, PK.

Step 1 (Modeling): Develop a model for the plant to be controlled $P=C\Phi B$, as well as bounds on our modeling error, in the following special form:

$$\dot{x}(t) = A\,x(t) + B\alpha(Cx(t)) + B\,u(t) \tag{4.54a}$$

$$y(t) = C\,x(t). \tag{4.54b}$$

Note that this model is in both controller and observer form. If our model is not orginally in this form, we must try to find a state transformation to bring it into this form. If the model does not fit this form, we can still attempt this procedure, but we will not be guaranteed the recovery step.

Step 2 (Specifications): Convert all available specifications into specifications on the loop broken at the plant output, or T=PK. Consult figure 4-1. We will call this the *desired loop* T_{des} Obviously, the more easily we can express our specifications in terms of T_{des}, the better off we will be. The specifications that can be easily translated in terms of T_{des} are

(a) Nominal Closed-Loop Stability. Obviously T_{des} must be closed-loop stable. In addition, since it will be realized as PK for some K, that realization must be internally closed-loop stable as well.

(b) Output disturbance specifications. If T_{des} is large for all signals in some set, then the response to output disturbances in that set will be small. Use the results of section 2.4 for calculations.

(c) Command following specifications. Similarly, by the results of section 2.4, if T_{des} is large for all signals in some set, then the error to commands in that set will be small.

(d) Robustness to unmodeled dynamics. We develop specifications for bandwidth, etc. and can express them in terms of T_{des} using the results of section 2.5.

The basic idea of this step is to capture *all* available specifications. Thus, if a loop T_{des} meets these specifications, we will be satisfied using it.

Step 3 (Target Loop Design): In this step we delovelop a target loop that will meet all of the specifications of step 2. In this variation of NMBC/LQR, we use a loop

$$T_{des} = C\Phi H \tag{4.55}$$

for some operator H. This gain H can come from many sources, as detailed in section 3.3. We want to make sure that our final closed-loop system will be stable, so we require that H produce a nondivergent estimator in the context of figure 4-3. One possibility that we know will be nondivergent if the system is detectable is the extended Kalman filter. Then we will have the following properties for the desired loop $T_{des} = C\Phi H$:

(a) Nominal Closed-Loop Stability. The filter loop is guaranteed to be closed-loop stable (corollary 3.6)

(b) Adjustable Performance. By varying the matrix Ξ in the EKF formulation, we can adjust the performance of the target loop T_{des}. For example, by making Ξ larger, we increase the performance and bandwidth. In the linear case, bandwidth control is many times sufficient to develop a reasonable "first pass" controller.

(c) Guaranteed Robustness. As shown in theorem 3.8, the loop $C\Phi H$ has many guaranteed properties, including good gain and phase margins, if we select Ξ large enough so that $H(t)$ does not vary much from trajectory to trajectory. Thus we can automatically obtain a fairly robust target loop.

Step 4 (NMBC Construction): We now construct the NMBC as shown in figure 4-3. The gain H comes from step 3, and the gain G must be chosen such that the closed-loop system is stable. The way to do this is to use the separation theorem of section 3.2 to realize that we should select G to be a stabilizing state feedback gain. This could be done in any of the ways discussed in section 3.4, however, because we will require that G have a special form to enable us to do loop recovery, we propose the folowing technique. We select our state feedback gain G as:

$$G(x) = G_\rho^{Lin} x + \alpha(Cx) \tag{4.56}$$

where G_ρ^{Lin} is a matrix defined by

$$G_\rho^{Lin} = \frac{1}{\rho} B^T K, \tag{4.57}$$

where K is the symmetric semi-positive definite solution to the Ricatti equation:

$$0 = A^T K + KA + Q - \frac{1}{\rho} KB^T BK. \tag{4.58}$$

If G is selected this way, it is clear that G stabilizes our plant (see section 3.4.5), and also has the asymptotic behavior

$$\lim_{\rho \to 0} \sqrt{\rho}\, G = WC, \quad W \text{ invertible}, \tag{4.59}$$

if the linear plant (A,B,C) is minimum phase.

Step 5 (Loop Operator Recovery): In this step, we apply theorem 4.2 and let ρ approach zero. We have that

$$\lim_{\rho \to 0} PK = C\Phi H = T_{des}, \tag{4.60}$$

if H is a fixed time varying matrix H(t), a constant matrix H, or a nonlinear operator H. However, in the EKF case, the trajectory of $H(t)$ depends on the internal state of the compensator, as $\Sigma(t)$ depends on the current compensator state z through $\nabla f(z(t))$. Thus, the recovery process (4.60) cannot work, as the recovered loop $C\Phi H$ does not have the same internal structure as each actual loop PK. Thus, while we do get a convergence to a limiting operator in the case of the EKF, this limiting operator, in general, is not $C\Phi H$. However, we can still use the recovery process if we realize that we had to have $H(t)$ relatively constant over different trajectories of the system in order that our guaranteed properties (theorem 3.10) held. Here, if $H(t)$ is relatively constant, then we can conclude that (4.60) holds and we can get approximate robustness recovery, with nondivergence guaranteed because we are using the EKF. As indicated in section 3.3.4, we can get this constancy property of $H(t)$ by choosing $\Xi = qBB^T$ with q large. Of course, if we use the CGEKF, or other constant gain observer, we do not have this problem, as H is constant.

If we now assume that we do get recovery, we then can select ρ sufficiently small so that our actual loop, PK, will approach the target loop, T_{des}, that we selected for its good characteristics. Furthermore, we know that the system will be closed-loop stable (including internally stable) for all values of $\rho>0$. Thus, for our selected value of ρ, we have designed a compensator K that stabilizes the plant P and meets all of the specifications.

Step 6 (System Simulation): Just as in the previous section, we must do many simulations to be sure that we have overlooked some important specification and to check the overall system performance. Again, if the system does not prove satisfactory, we must go back to steps 2 and 3.

In section 5.4, we demonstate this recovery at the plant output procedure on a simple nonlinear pendulum model.

End of Design Procedure

4.3.4 Formal Loop Shaping

We now give the steps for designing a controller based on the Formal Loop Shaping method [2], using theorem 4.3. This method is most applicable if the specifications can most easily be poes in terms of the loop broken at the plant output, as in the previous method, and if the plant is open-loop incrementally stable.

Step 1 (Modeling): Develop a model for the plant to be controlled, $P=C\Phi B$, and bounds on the modeling error. The plant should be incrementally stable.

Step 2 (Specifications): Convert all available specifications into specifications on the loop broken at the plant output, or $T=PK$. Consult figure 4-1. We will call this the *desired loop* T_{des}. Obviously, if the more easily we can express our specifications in terms of T_{des}, the better off we will be. The specifications that can be easily translated in terms of T_{des} are discussed in the previous section, step 2. Again, here idea is to capture *all* available specifications, so that if a loop T_{des} meets these specifications we will be satisfied using it.

Step 3 (Target Loop Design): In this step we delovelop a target loop that will meet all of the specifications of step 2. In this variation of NMBC/LQR, we can use essentially any loop we wish. We select a system of the form (4.39)

$$T_{des} = D\Psi H, \tag{4.61}$$

where $\Psi H[I+D\Psi H]^{-1}$ must be stable. Since we are not constrained as in the previous two procedure descriptions, we can place any dynamics we wish in Ψ. One possible choice would be a linear loop, designed with either a Linear-Quadratic-Regulator loop, or a Kalmn filter loop. Then we would have

(a) Nominal Closed-Loop Stability. The regulator and Kalman filter loops are guaranteed to be closed-loop stable, and thus so is $\Psi H[I+D\Psi H]^{-1}$.

(b) Adjustable Performance. We can easily adjust the parameters in the formulation of these optimaization problems to make the target loop T_{des} look like just about anything.

(c) Guaranteed Robustness. The LQ and KF loops have many guaranteed properties [6], including $(1/2,\infty)$ gain margin and +/-60 degrees phase margin. Thus we automatically have obtained a robust target loop.

Step 4 (NMBC Construction): We now construct the NMBC shown in figure 4-4, using the parameters that we have determined in the previous step. Note that we will require that the closed-loop system be stable. To insure this, we use the result of Q-parameterization from theorem 4.4, which says that our system will be closed-loop stable if

(a) $[I+G_x\Phi B]^{-1}$ stable,

(b) $G_s\Psi H[I+D\Psi H]^{-1}$ stable, and

(c) P is incrementally stable.

This can be seen by comparing figures 4-4 and 4-5, and applying theorem 4.4. In addition, to do the LQR step, we will need to find $G_{s,\rho}$ and $G_{x,\rho}$ such that

$$\lim_{\rho\to 0} \sqrt{\rho}\, G_{x,\rho} = WC \tag{4.62}$$

and

$$\lim_{\rho\to 0} \sqrt{\rho}\, G_{s,\rho} = WD. \tag{4.63}$$

One way that this can be done is to formulate an optimal control problem as follows. For the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{s}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) \\ f_1(x(t)) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \tag{4.64}$$

find u(t) to minimize the cost functional

$$J(x_0, s_0, u(\cdot)) = \frac{1}{2}\int_0^\infty [\ |Cx(t)+Ds(t)|^2 + \rho\ u^T(t)u(t)\ ]\ dt. \tag{4.65}$$

The solution will be expressable in the form

$$u = -\ g_x(x) - g_s(x) = -\ G_x x - G_s s. \tag{4.66}$$

By theorem 3.17, we can conclude that condition (a) above holds, and using theorem 3.18 we can conclude (4.62) and (4.63) hold if certain conditions analogous to the minimum phase conditions of linear systems can be verified (see section 4.5).

Remark By using the trick of section 4.3.3, where we pick $u=-G_\rho^{Lin}x-\alpha(x)$ for our plant in controller form and make the state feedback system linear, and by choosing $D\Psi H$ linear, we can make the minimization problem (4.64-4.65) a Linear-Quadratic-Regulator problem, which is easily solved.

Step 5 (Loop Operator Recovery): In this step, we apply theorem 4.1 and let ρ approach zero. We have that

$$\lim_{\rho\to 0} PK = D\Psi H = T_{des}. \tag{4.67}$$

If we select ρ sufficiently small, then our actual loop, PK, will approach the target loop, T_{des}, that we selected for its good characteristics. Furthermore, we know that the system will be closed-loop stable (including internally stable) for all values of

$\rho>0$. Thus, for our selected value of ρ, we have designed a compensator K that stabilizes the plant P and meets all of the specifications.

Step 6 (System Simulation): In order to be sure that all specifications were in fact included in step 2 and that nothing was neglected, we must do complete simulations, as in the previous procedures.

End of Design Procedure

4.4 Design Hints

This section is an informal presentation of various ideas that tend to be helpful in linear multivariable control design, and thus will most certainly be useful for nonlinear control design. The following two sections discuss scaling of states, inputs, and outputs, and augmentation with additional dynamics.

4.4.1 Scaling

This section will discuss the scaling of system variables to make the system easier to deal with. One might scale a model of the plant to be controlled in order that the controller design process goes more smoothly and encounters fewer numerical difficulties due to ill-conditioning. One might also scale the states of a controller that had already been designed so that its implementation could be done with fewer significant digits in fixed-point arithmetic, or scale its inputs or outputs to make them compatible with the interface equipment. Since there are many additional reasons for scaling variables, we now consider the scaling process.

Consider a system

$$\dot{x}(t) = f(x(t)) + B\,u(t) \tag{4.68a}$$

$$y(t) = C\,x(t), \tag{4.68b}$$

where the states, inputs, and outputs may have vastly different magnitudes. We can choose a new set of scaled variables, u_s, x_s, and y_s by choosing the scaling matrices S_u, S_x, S_y so that the new state variables

$$y_s = S_y^{-1}\,y \tag{4.69}$$

$$x_s = S_x^{-1}\,x \tag{4.70}$$

$$u_s = S_u^{-1}\,u \tag{4.71}$$

each have components with roughly equal magnitudes. For illustration purposes, we will now scale the variables so that each component has a maximum magnitude of unity. We simply select

$$S_y = \text{diag}\left\{ \max\{y_1\},\ \max\{y_2\},\ldots,\ \max\{y_m\} \right\} \tag{4.72}$$

$$S_x = \text{diag}\left\{ \max\{x_1\},\ \max\{x_2\},\ldots,\ \max\{x_n\} \right\} \tag{4.73}$$

$$S_u = \text{diag}\left\{ \max\{u_1\},\ \max\{u_2\},\ldots,\ \max\{u_m\} \right\}. \tag{4.74}$$

where by "max" we mean an approximate estimate of an upper bound for the magnitude of that variable. The transformed system is

$$\dot{x}_s(t) = S_x^{-1} f(S_x x_s(t)) + S_x^{-1} B S_u u_s(t) \tag{4.75a}$$

$$y_s(t) = S_y^{-1} C S_x x_s(t). \tag{4.75b}$$

The advantage of using this new tranformed system is that all our stability, performance, etc. criterion become less conservative, as we always treated each component of, say x, equally. If one was in tons and another in ounces, our bounds would become less useful. Thus scaling all variables allows us to have a common reference point to analyze and design for the system.

It is also possible to "scale" variables nonlinearly, i.e. a nonlinear change of coordinates. One reason to do this is to try to make the system "nicer" (read linear); the transformation designs discussed in chapter 3 touch on this subject.

This is really all there is to scaling. It is very simple in concept, yet in practice it is very difficult. The main difficulty lies in deciding what the maximum values are. At times this involves making value judgements about the relative importance of one variable versus another. It frequently requires several iterations to reach a reasonable scaled system, but in the end having such a scaled system makes life much easier.

4.4.2 Dynamic Augmentation

In this section we shall briefly discuss the process of *dynamic augmentation*, or the addition of certain dynamics to produce desirable effects in the final closed-loop system. We will present an example showing the introduction of free integrators to reduce steady state errors, although other dynamics might be desirable depending on the circumstances.

Suppose we start with a plant model like (4.68) and we wish to design a controller to make the closed-loop system have zero steady-state error to input disturbances, w, when we set r,d=0 in figure 4-1. Suppose further that the system has no free integrators in it, i.e. f^{-1} exists. Then we add integrators to the output of the system, and define a new output y_p as

$$\dot{y}_p := y. \tag{4.77}$$

We now use the following as our design plant model:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}_p(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) \\ Cx(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \tag{4.78a}$$

$$y_p(t) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ y_p(t) \end{bmatrix}. \tag{4.78b}$$

We now apply the loop recovery at the plant input procedure, described in section 4.3.2, so that we can shape the loop (-K)(-P) by first shaping the target loop GΦB. One possibility would be to use an optimal control formulation to obtain a good function G, which has inherent robustness properties. From theorem 3.17, the "L_2 domain inequality", we have

$$\rho \frac{\|G\Phi B\|_{2,\tau}}{\|u\|_{2,\tau}} + \rho \geq \rho \frac{\|[I+G\Phi B]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq \left[\rho + \frac{\|C\Phi Bu\|^2_{2,\tau}}{\|u\|^2_{2,\tau}} \right]^{1/2} \qquad (4.79)$$

where Φ,B, etc. are the design plant matrices. If we pick $m(x)=x^TC^TCx$, we get the integrator action in $G\Phi B$ (since it is in $C\Phi B$). We can see this because when $C\Phi B$ is large, $G\Phi B$ is also large (we can neglect the ρ terms). Thus we can shape the loop $G\Phi B$ to our liking and use the LOR procedure to make certain that the actual loop (-K)(-P) approached the target loop $G\Phi B$.

In our final NMBC, for constant disturbances we will have y going to zero. This is because if it did not go to zero, then y_p would blow up, which it cannot do, because of the guaranteed stability. This zero steady state error is guaranteed even if we do not use the LOR procedure. We only require that the closed-loop system be stable and thus that we incorporate a nondivergent estimator into our compensator.

There are many variations possible on this theme. We could add integrators at the input of the plant to improve low frequency command following performance, or we could add resonant dynamics at a particular frequency to improve disturbance rejection at that frequency, for example 60 cycle hum in certain critical applications.

We now conclude this section with the thought that since the NMBC/LOR methodology is still in its infancy, when examples are tried, many new tricks may be uncovered. There are still many linear design tricks which seem likely to have their counterparts in the nonlinear world. One of the more promising relations seems to be the "L_2 domain inequality" of theorem 3.17, which may lead the way to loop shaping ideas, as will be mentioned in section 6.2.3.

4.5 Nonminimum Phase Systems

This section will discuss the idea of minimum phase and its relationship to the theory presented in this thesis. Recall that in the linear case a minimum phase system is one that has all of its zeros in the left-half plane, or equivalently, has a stable inverse. We would like to make a similar definition for the nonlinear situation, except that we run into a problem. Consider the inverse of a nonlinear system P, formally written P^{-1}. Now, in general P^{-1} will not be "proper", in that it will do some pure differentiations on input signals. This causes problems, for consider a signal that moves up in frequency indefinitely, for example

$$w(t) = A_o \sin(t\omega(t)),$$

where

$$\omega(t) = t^2.$$

If we apply this to P^{-1}, we see that we will have a growing output $P^{-1}w$, due to the differentiation. Put another way, to produce $w=Pu$, we must have u growing without bound. The result of all this is that the operator P^{-1} is unstable by our definition. Note that we have a similar problem with linear systems, as P^{-1} is not, say, L_∞-stable either (impulses are not in $\mathcal{L}_\infty$), but we get around this by just considering the poles of P^{-1}, or equivalently, the zeros of P. In the nonlinear case, we do not have the luxury of being able to check zeros easily (although see [64] and below). Two possible ways to still retain the idea of stable inverse are:

(a) Only allow signals with no frequency component above some cutoff frequency when considering the minimum phaseness of a system.

(b) Consider the stability of XP^{-1} or $P^{-1}X$, where X is a linear system with n poles and no zeros, e.g.

$$X = I\cdot\frac{1}{(\tau s+1)^n}$$

for some fixed $\tau>0$.

The reason that we care about minimum phaseness is that in the linear case, we must have a minimum phase plant in order to obtain the loop transfer recovery. The reason that this is the case is fairly straightforward. Since the recovery procedure involves the approximate inversion of the plant and right-half plane zeros cannot be cancelled in a stable manner, we have a problem. Note that the reason recovery works for open-loop unstable plants is that the dynamics of the recovered loop (say $G\Phi B$) are the same as the plant ($C\Phi B$), so that the open-loop poles do not really get cancelled. We could have recovery for minimum phase plants if we selected the target loop to have the same "zeros" in the right-half plane as the plant (See [2] for the linear case).

Although we did not mention the minimum phase assumption in previous sections of this chapter, it was already built into one of the key assumptions: the asymptotic behavior of H_μ and G_ρ. In the linear case, the results of [63] indicate that we will get the correct asymptotic behavior of G_ρ only if the orginal plant is minimum phase. The limit V_0 of theorem 3.18 is zero for minimum phase plants, and thus

$$\lim_{\rho \to 0} \sqrt{\rho}\, G = WC. \tag{4.82}$$

Therefore, we could make the following tentative definition.

Definition A plant CΦB is minimum phase if there exists a set of state feedback functions, G_ρ, parameterized by ρ, such that (4.82) above holds, and G_ρ stabilizes the plant for all $\rho > 0$.

In [64] a alternate definition for minimum phaseness is given for relative degree one nonlinear systems which seems to be related. The definition involves transforming the nonlinear system in such a way that the *zero dynamics* are exposed. They define the system to be minimum phase if the zero dynamics are stable and show that minimum phase systems can be stabilized by high gain feedback. Upon preliminary investigation it appears that this definition is equivalent, with suitable technical assumptions, to our notion above of a stable inverse.

We would like to use one of these definitions to show that recovery works if and only if the plant is minimum phase, but it appears that this may not be necessary. If we take a lesson from linear theory, we realize that right-half plane zeros put performance restrictions on systems, independent of the methodology used to design the controller [65]. In the LQG/LTR methodology, it is recommended that the LTR procedure be tried even when dealing with a nonminimum phase system. This forces the controller to find the "best" stable inverse to the plant and substitute the target loop dynamics. Thus poor recovery (performance) is obtained in certain frequency ranges, but since we have right-half plane zeros, we have to settle for reduced performance anyway.

Thus we recommend for the NMBC/LOR methodology that one lets $\rho \to 0$ in whichever algorithm one is using to generate G_ρ (optimal control, tranformation methods, etc.). Simulations will determine whether or not the recovery has taken place, and for which signals. Note that for the transformation methods, minimum phase behavior is determined by the minimum phaseness of a linear system and thus can be easily checked. Similar remarks hold for the recovery at the plant input and formal loop shaping procedures.

4.6 Critique

In this section we discuss the NMBC/LOR loop shaping philosophy relative to other possible schemes on a sound basis. We will attempt to discuss informally the robustness of different types of designs, using loop operators, which, as we saw in chapter 2 are the relevant quantity in unstructured robustness analysis.

This thesis has been concerned with the design of a compensator K so that the closed-loop system of figure 4-1 has good properties.

In particular, we were concerned about either the loop broken at the plant input, (-K)(-P), or the loop broken at the plant output, PK, and we were able to design either one of them to have good robustness properties, with some restrictions. Thus we were able to design within the bandwidth constraints that every physical system has.

Let us now consider a different design strategy, which we will call the *heuristic method*. Suppose that we could design some sort of feedback that would linearize our plant. We might do this for a plant in controller form by measuring or estimating accurately the states needed to linearize the plant in the framework of section 3.4.5. Then we could apply linear methodologies to the linearized plant. We show a generalized version of this idea in figure 4-6. Note that we could consider it a two-step compensation process, where the first step consists of an inner loop compensator, K_1, which makes P linear, i.e. so that

$$P[I+K_1P]^{-1} \tag{4.83}$$

is linear. The second step consists of an outer loop compensator, K_2, using, say, LQG/LTR, so that we have good loops broken at either the "input", point (i), or the "output", point (ii). We use quotes here because they are *not* the same loop breaking points that we *really* care about for robustness, which are either at the plant input (iii) or the plant output (iv). This is an inherent problem with the

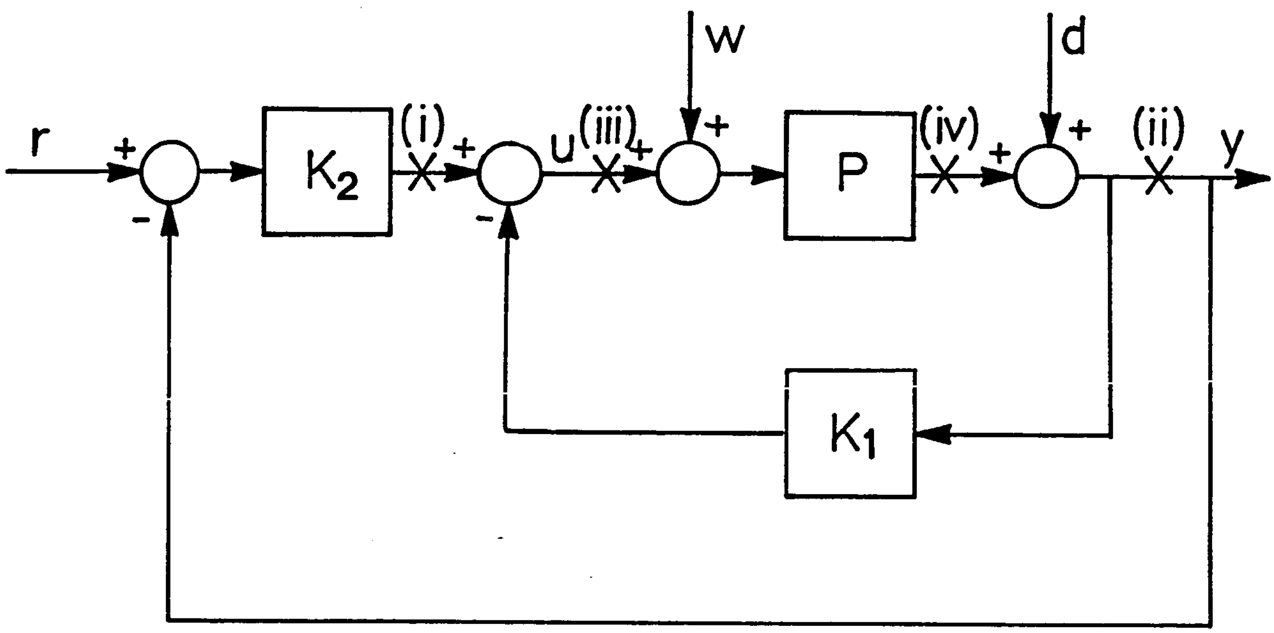


Figure 4-6: Two-Step Compensation

two-step compensation process, and with any other procedure, where the loops that can be molded are not the same as the "true" loops that we wish to design.

As a more concrete example, we might design K_2 so that we got excellent command following performance in figure 4-6. This would be done by shaping the loop broken at point (ii). We might even be careful not to exceed what we thought were the bandwidth constraints. But what is the true bandwidth? The actual loop that is important for robustness is the loop broken at point (iv), which is not easily related to the design loop (ii). We might have a much faster loop at (iv) than at (ii) and thus all our efforts to design well at (ii) did not help us at all at point (iv).

Note the the external linearization methods [21,22,23,24,25,26] are of the above two-step compensation procedure. However, as we pointed out earlier in this chapter (throughout section 4.3), there are ways to use their results in the NMBC/LOR framework to achieve a single-loop design that does not suffer from the above drawbacks.

The above is one of the strong points of NMBC/LOR, namely that it can be used to control the loops at the true plant inputs and outputs. Specifically, it can be used to make them robustly closed-loop stable and have the desired performance. However, it does have some restrictions when we are concerned with shaping the loop at the plant output. For example, recovery at the plant output requires that our plant be in both controller and observer form, which is very restrictive. The formal loop shaping (FLS) procedure allows us to arbitrarily shape the loop at the plant output but guarantees closed-loop stability (for the time being) only for incrementally stable plants.

CHAPTER 5. A NUMERICAL SIMULATION

5.1 Introduction

In order to demonstrate some of the details of the NMBC/LQR methodology, we present the results of a very simple numerical simulation. Using a model of a damped swinging pendulum, we attempt to illustrate the following:

(a) Convergence of the estimation error as the extended Kalman filtering noise parameter μ goes to zero, in a demonstration of theorem 4.1

(b) Recovery at the plant output.

(c) The guaranteed gain margins of the EKF and recovered loops.

(d) A "frequency sweep" technique for analyzing sensitivity functions.

(e) Use of the "frequency sweep" technique to show the properties of the EKF and recovered loops.

5.2 Plant Model and Compensator

For the purposes of this simulation, we selected a simple damped pendulum model:

$$\dot{x} = f(x) + B\,u \tag{5.1a}$$

$$y = C\,x \tag{5.1b}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{5.2a}$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{5.2b}$$

where the variables are

x_1 = angle of pendulum, in radians, with zero chosen as down.

x_2 = angular velocity of pendulum, in radians/second.

u = input torque to pendulum.

We do not restrict the states in this model, so that for example, $x_1=3\pi$ is pointing straight up, after going around one full revolution. Note that this makes our model unstable for inputs with magnitude greater than unity, as the torque is high enough to cause the pendulum to keep spinning around, and thus, for $x_1(t)$ to be unbounded. The linearization of the model at the origin ($x_1=0$) gives poles at

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}j \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}j,$$

with a magnitude of 1.0 rad/sec and a damping coefficient of 0.5.

We next consider the compensator that we will use. Note that our model above (5.1) is in both controller and observer form. Thus we decided to use a transformation state feedback controller, as discussed in section 3.4.5. Since we can place the closed-loop poles arbitrarily (the transformation controller gives linear closed-loop dynamics), we decided to parameterize them by $\rho>0$ and place them on a

Butterworth pattern. We chose as our state feedback function $u=-g_\rho(x)$:

$$g_\rho(x) = \sin(x_1) + x_2 + \left[\frac{1}{\rho}\right]^{1/2} x_1 + \left[\frac{1}{\rho}\right]^{1/4} x_2. \tag{5.3}$$

We chose this function as it has the following properties:

(a) The closed-loop system with $u=-g_\rho(x)$ is stable for $\rho>0$.

(b) Asymptotic convergence:

$$\lim_{\rho\to 0} \sqrt{\rho}\, g_\rho(x) = x_1 = Cx. \tag{5.4}$$

(c) For each $\rho>0$, there exists a k_ρ such that

$$|\nabla g_\rho(x)| \le k_\rho \quad \forall\ x\in\mathbb{R}^2. \tag{5.5}$$

We need the properties (a) and (c) to be able to use the separation theorem (theorem 3.1) of section 3.2, while property (b) will be needed to use the recovery theorem (theorem 4.2) of section 4.3.3.

Turning to the state estimator design, since our plant (5.1) is in observer form (section 3.3.6), we could use a transformation based observer technique (section 3.3.6) to design a nondivergent observer, but with no other priori guarantees. Thus, instead we chose to use an extended Kalman filter, with its guaranteed robustness properties (section 3.3.4). As presented in section 3.3.3, there is really only one design parameter for the EKF, namely the symmetric positive semi-definite matrix Ξ. The other parameters are Σ_0, the intial condition for the covariance, and the intial time t_0 for the covariance propagation to begin, i.e. $\Sigma(t_0)=\Sigma_0$. We simply chose $\Sigma_0=0$, and $t_0=-4$ seconds. This was determined by simulation to be enough time for the covariance equation to reach roughly steady state for the linearization point x=0. Remember that we do not start the state equations (5.2) until the EKF has had sufficient time to intialize itself (section 3.4.3). This does not mean that if the EKF happens to encounters noise and drifts from the true state that it will need to be reinitialized; its nondivergence is guaranteed by theorem 3.6. It is more a matter of not starting at a bad Σ_0 than having to select the "correct" Σ_0.

We selected the design parameter $\Xi=qBB^T$, where $q>0$. We will think of q as $1/\mu$ in the recovery theorem 4.1. Note that since Ξ is not full rank, we cannot guarantee the nondivergence of the EKF by theorem 3.6 unless we can check the uniform controllability of $[\nabla f, B]$. It was decided to just go ahead without that theoretical justification because it was felt that (a) the system was intuitively controllable through B, and (b) it would be useful to show that the EKF could be used successfully without a lot of technical restrictions having to be checked for each case.

Since we have designed a nondivergent estimator and a stabilizing state feedback function, we can now put them together by choosing $u=-g_\rho(\hat{x})$, and utilizing the NMBC/LOR structure depicted in figure 4-3.

The following sections are now devoted to the simulation of this NMBC/LOR closed-loop system, in order to demonstrate its various properties.

5.3 Convergence of Estimation Error

As proved in theorem 4.1 (recovery at the plant input), when r and d are zero, then $\hat{x} \to x$ (since z is $\hat{x}$ when r=0) as $\mu \to 0$ in any nondivergent estimator where the gain H_μ obeys

$$\lim_{\mu \to 0} H_\mu \sqrt{\mu} = BW \tag{5.6}$$

for some invertible W. Then it becomes clear that the rest of theorem 4.1 holds, namely, that

$$\lim_{\mu \to 0} (-K_\mu)(-P)(u+w) = \lim_{\mu \to 0} (-Gz) = -Gx = -G\Phi B(u+w) \tag{5.7}$$

where u+w is the input to the plant in the structure of figure 4-1.

We now demonstrate this convergence. With u,d,r=0, we simulate the step response in w. Thus the estimator obtains information about this step only through observation of the output of the plant, y. The simulation is shown in figures 5-1 and 5-2 for various values of q, where $\Xi = qBB^T$. Figure 5-1 contains the state estimates (since r=0, $\hat{x} = z$) for a 0.80 magnitude step in w, while figure 5-2 shows the estimates for a 1.20 magnitude step in w. Note that we get convergence of the state estimates to the actual states as $q \to \infty$. Since the estimator is presumably nondivergent, we expect that the error between the actual states and the estimates should be bounded in time for each value of q (because the input w is), and we can see that it is. This is true even when the response of the states is unbounded, as in figure 5-2. As noted in the previous section, we expect to see unbounded behavior for step inputs to the plant larger than unity in magnitude.

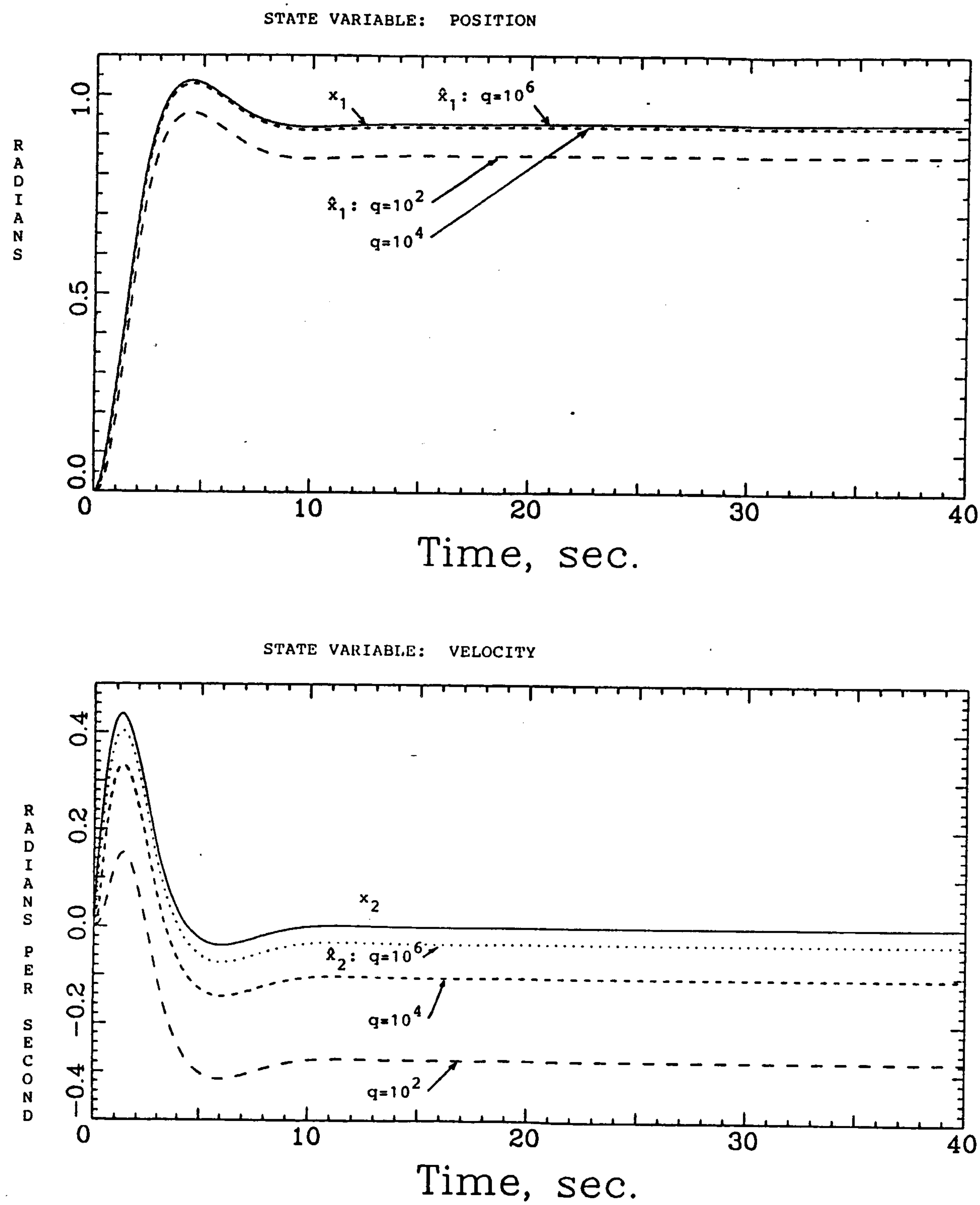


Figure 5-1: State Estimation: $w(t)=0.80u_{-1}(\cdot t)$

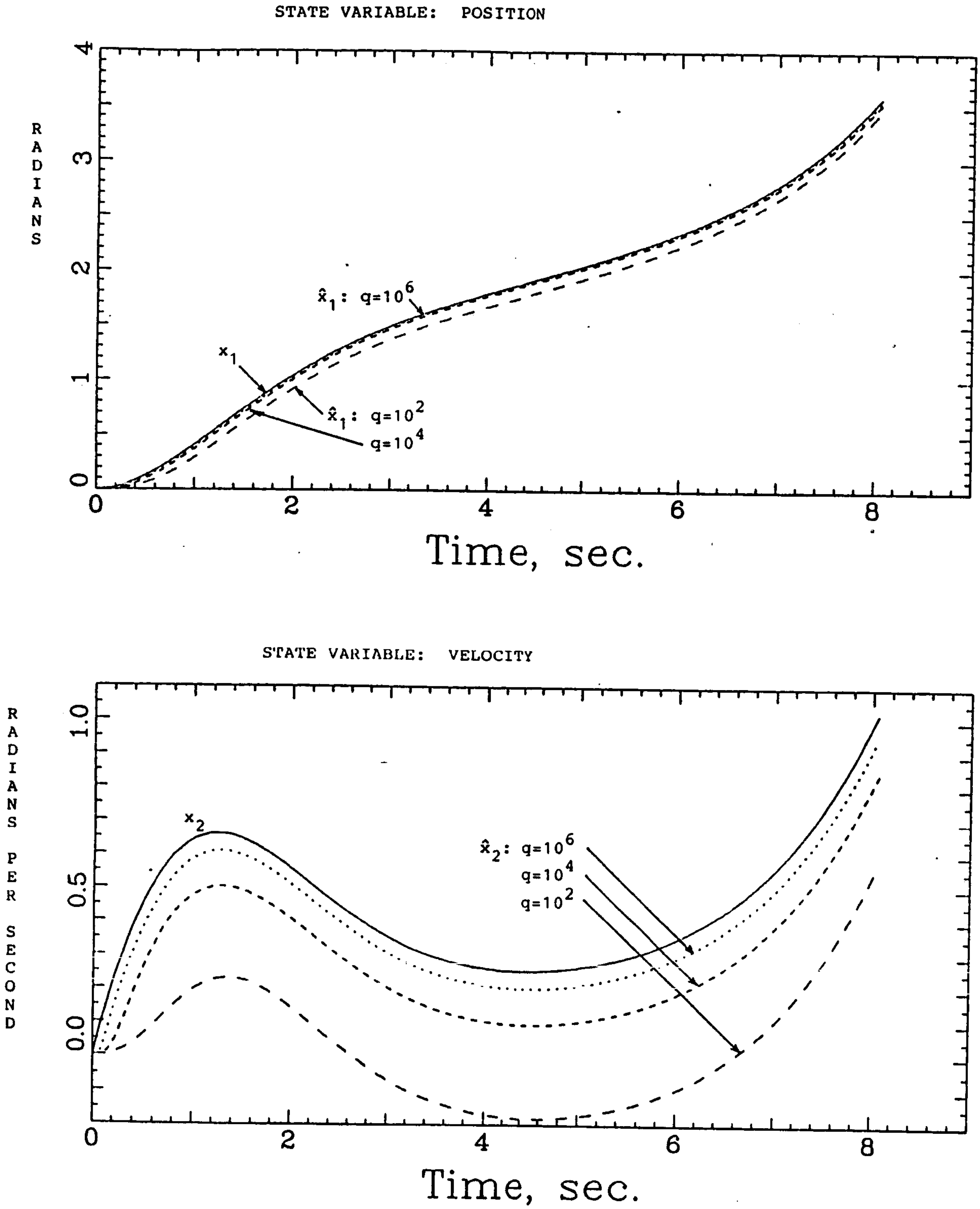


Figure 5-2: State Estimation: $w(t)=1.20u_{-1}(t)$

5.4 Recovery at the Plant Output

We now demonstrate the recovery procedure at the plant output. Our plant and compensator are already set up in a form suitable for us to use theorem 4.2, i.e., the plant is in both controller and observer form, the state feedback linearizes the plant, and we have the asymptotic behavior (5.4) that we require.

Figures 5-3, 5-4, 5-5, and 5-6 show the step responses of the closed-loop system for step magnitudes of 0.10, 1.00, 3.14, and 4.71, respectively. In each plot, we can see that as $\rho \to 0$, the closed-loop step responses converge to the closed-loop step response of the $C\Phi H$ loop, i.e. letting $r(t)=Au_{-1}(t)$ be the step response, we see

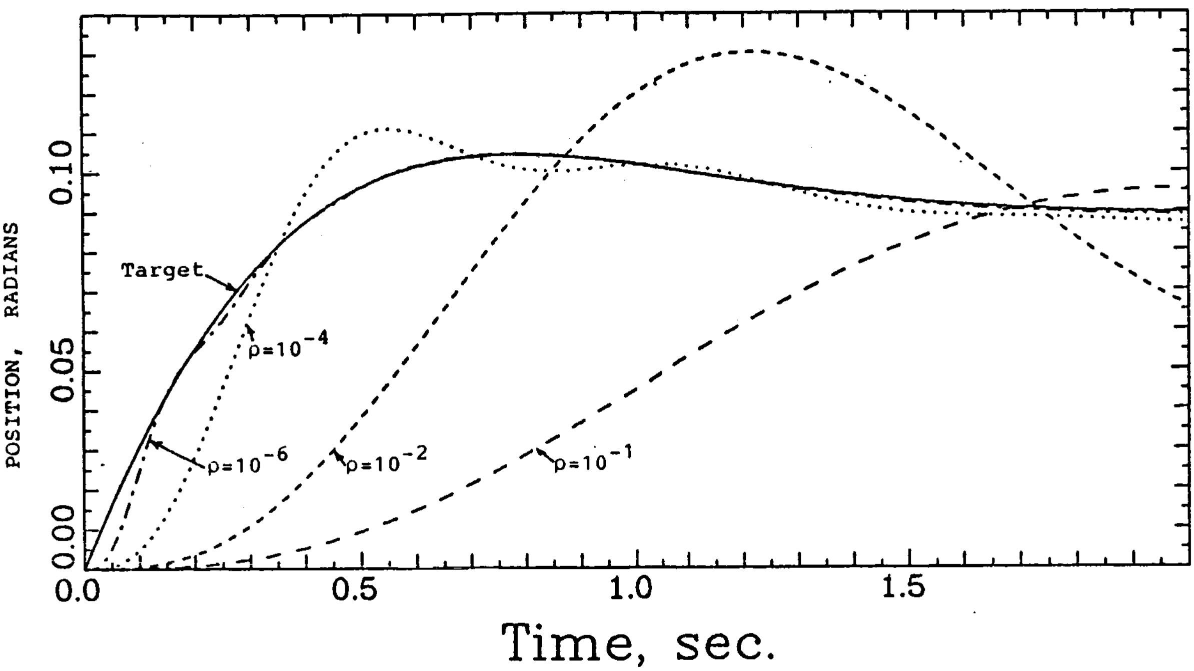
$$PK_{\rho}[I+PK_{\rho}]^{-1} \rightarrow C\Phi H[I+C\Phi H]^{-1} \quad \text{as } \rho \to 0. \tag{5.8}$$

Note that the shape of the responses for different input magnitudes are very close to each other (obviously, in a linear system they would all be scaled versions of each other), which indicates that we are using a fairly large gain for $C\Phi H$. This is good because large gains ensure the constancy of $H(t)$ which helps in two ways: (1) recovery is guaranteed (see section 4.3.3, step 5), and (2) the robustness properties of the EKF loop ($C\Phi H$) are ensured (section 3.3.4, theorem 3.11).

We now verify one additional property of the recovery at the plant output theorem 4.2, namely, that we expect

$$Cz = z_1 \to 0 \quad \text{as } \rho \to 0. \tag{5.9}$$

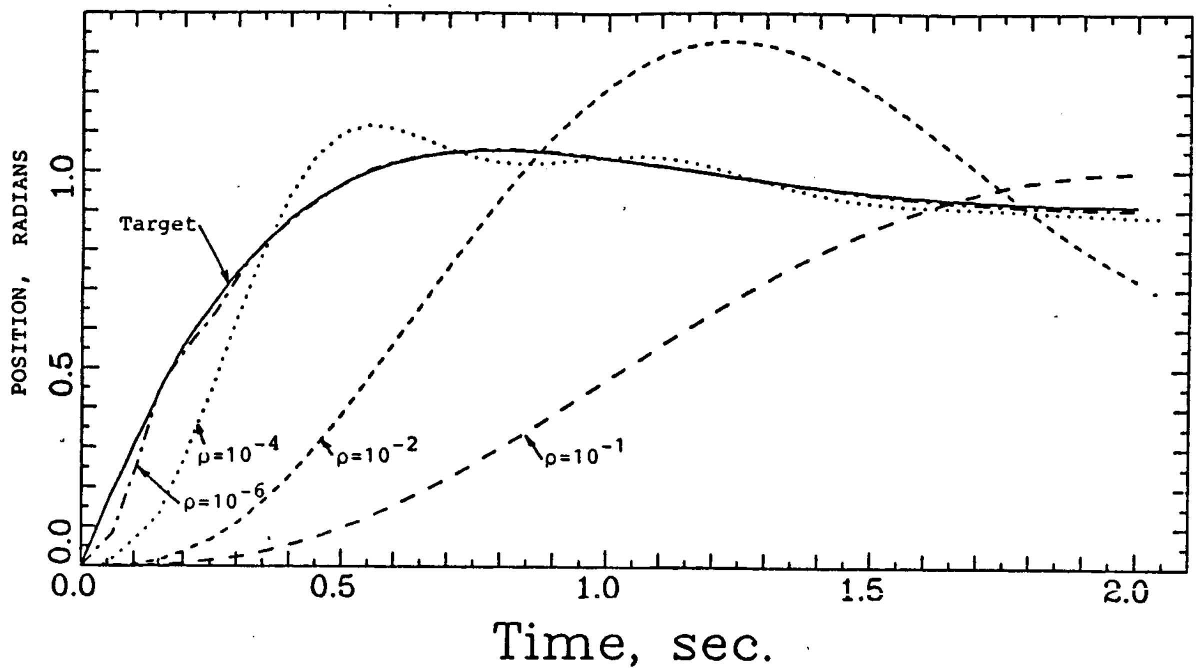
We show the compensator state, z, in figure 5-7, for the particular step response with magnitude 1.00 (corresponding to figure 5-4). Note that we do have $z_1 \to 0$ as $\rho \to 0$.



Target: $C\Phi H[I+C\Phi H]^{-1}r$

Values of ρ: $PK_\rho[I+PK_\rho]^{-1}r$

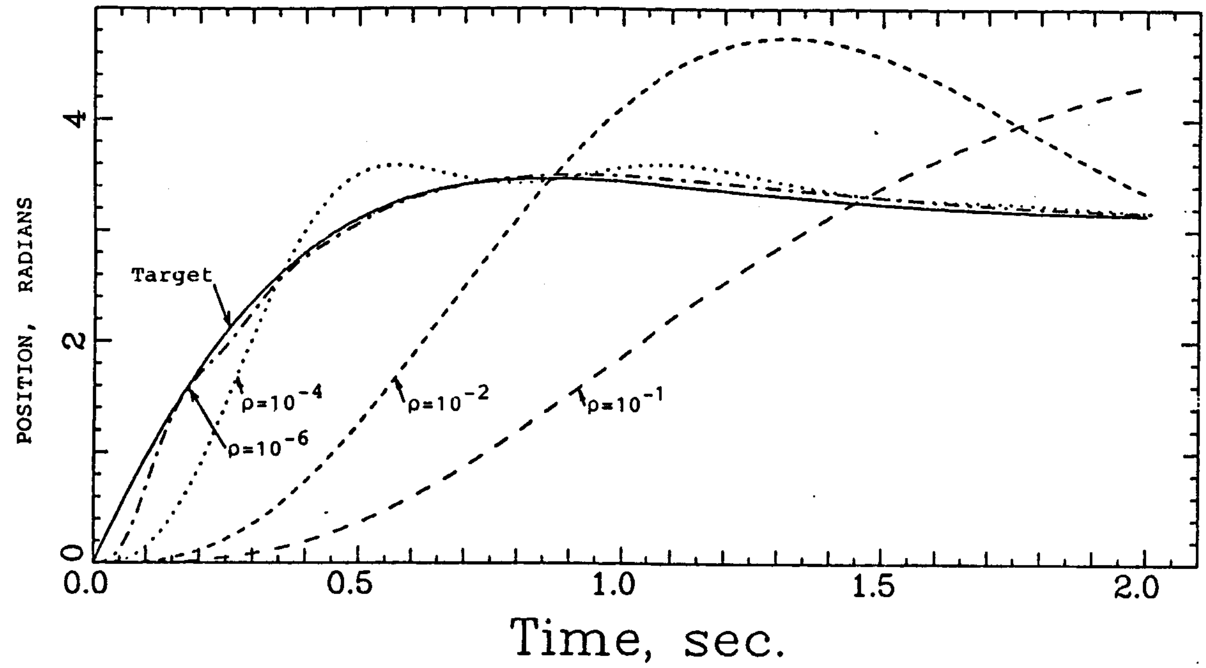
Figure 5-3: Recovery at the Plant Output: $r(t)=0.10u_{-1}(t)$



Target: $C\Phi H[I+C\Phi H]^{-1}r$

Values of ρ: $PK_{\rho}[I+PK_{\rho}]^{-1}r$

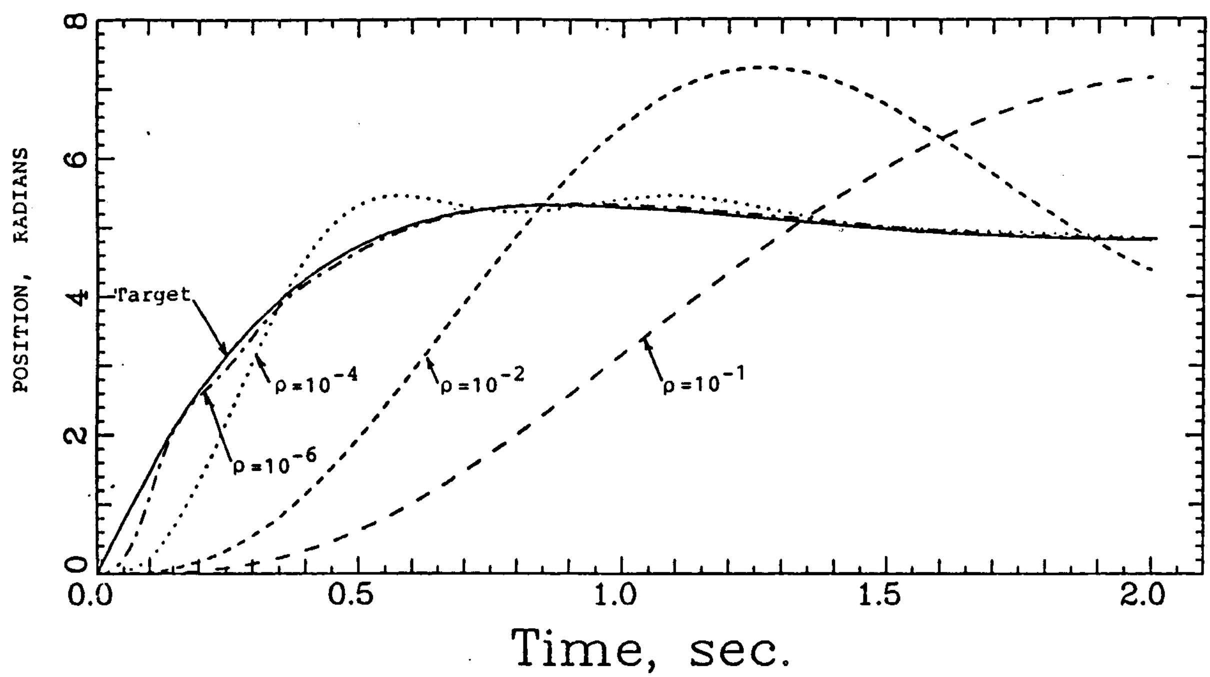
Figure 5-4: Recovery at the Plant Output: $r(t)=1.00u_{-1}(t)$



Target: $C\Phi H[I+C\Phi H]^{-1}r$

Values of ρ: $PK_{\rho}[I+PK_{\rho}]^{-1}r$

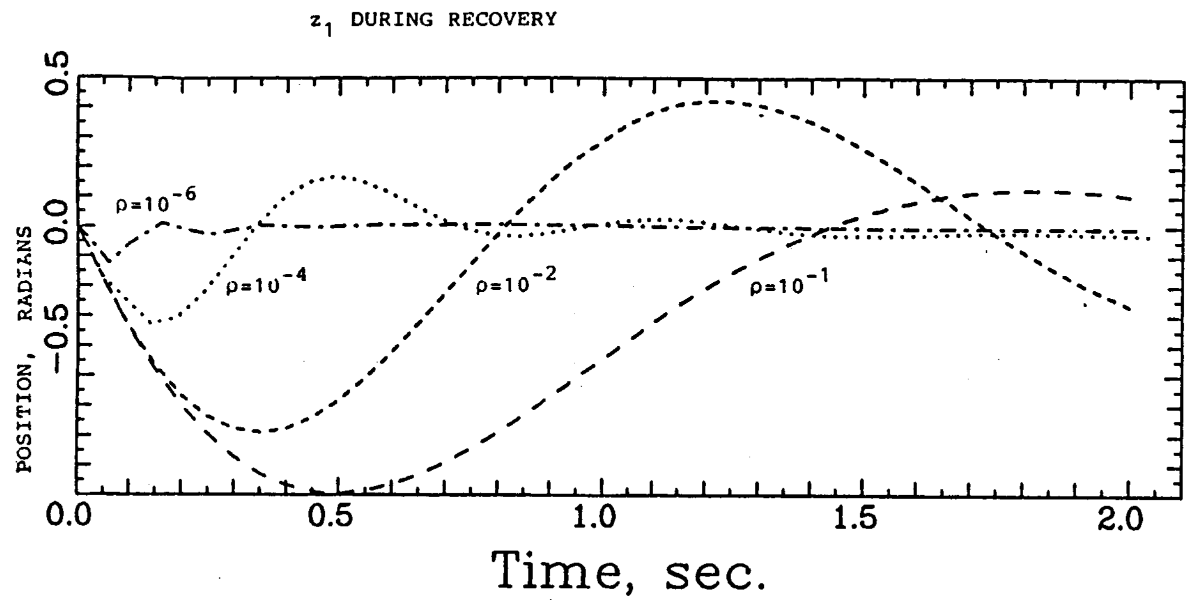
Figure 5-5: Recovery at the Plant Output: $r(t)=3.14u_{-1}(t)$



Target: $C\Phi H[I+C\Phi H]^{-1}r$

Values of ρ: $PK_\rho[I+PK_\rho]^{-1}r$

Figure 5-6: Recovery at the Plant Output: $r(t)=4.71u_{-1}(t)$



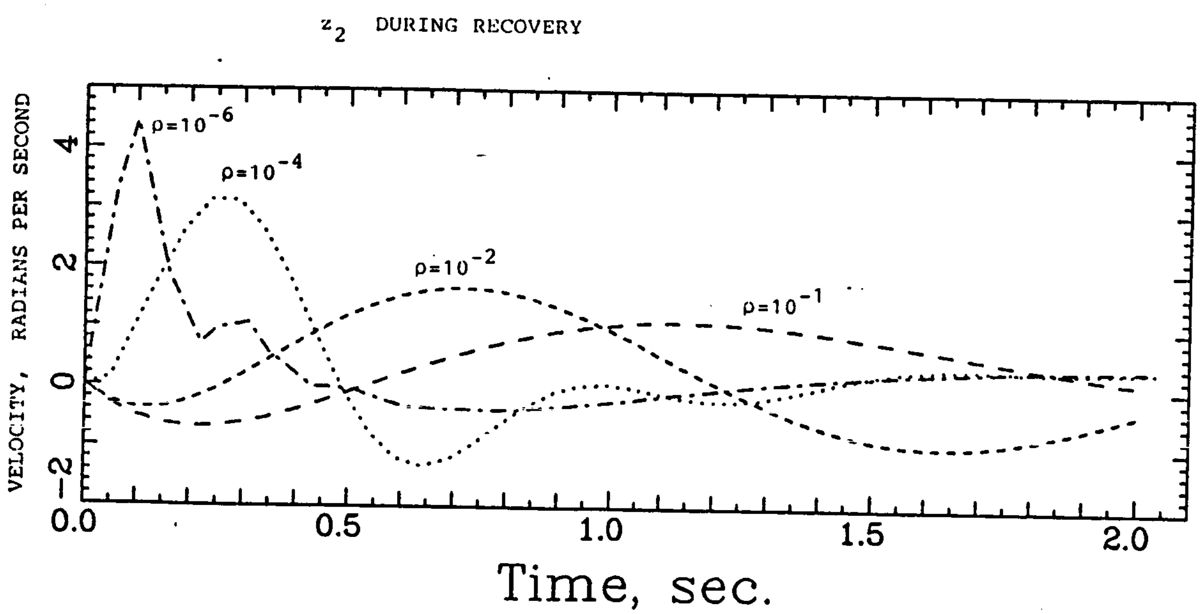
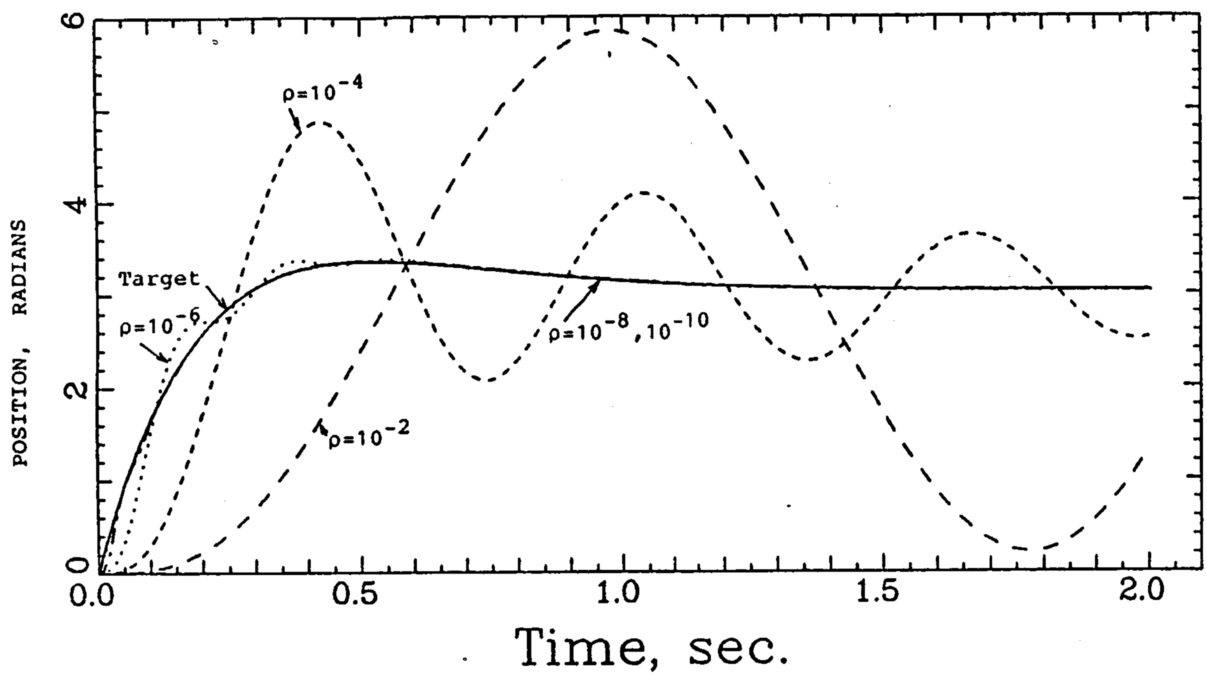


Figure 5-7: Compensator States: $r(t)=1.00u_{-1}(t)$

5.5. Gain Margin Tests

We now check the robustness of our closed-loop system to gain variations. Theorem 3.11 tells us that if we have a sufficiently constant H(t), then we will have an infinite upward gain margin. As discussed in section 5.2, since we selected a fairly high q, we expect that H(t) is relatively constant. To check this, we perturbed our plant by scaling the output by the factors of 2.0, 5.0, and 10.0. the respective step responses are shown in figures 5-8, 5-9, and 5-10 for a step input magnitude of 3.14, corresponding to figure 5-5. We see in these plots that (1) the target loop $C\Phi H[I+C\Phi H]^{-1}$ is stable for all the gains 2, 5, and 10, and (2) the actual loops $PK_{\rho}[I+PK_{\rho}]^{-1}$ approach the target loop as $\rho \to 0$, and thus approach the target loop in their gain margins.

Consider first figure 5-8. Here we have one closed-loop response that is unstable (or marginally so), the one corresponding to $\rho=10^{-2}$. The others are all "closer" to the target loop, and have a higher gain margin. In figure 5-9, with a gain perturbation of x5, the responses for the cases $\rho=10^{-2}$ and $\rho=10^{-4}$ go unstable, while the responses for the systems with better recovery remain stable (although the case $\rho=10^{-6}$ is starting to be oscillatory). Finally, in figure 5-10, with a gain factor of 10, the case $\rho=10^{-6}$ also goes unstable. However, the cases $\rho=10^{-8}$ and $\rho=10^{-10}$ are still stable. This is because, roughly, (thinking of linear systems) they have recovered sufficiently so that they match the target loop over the

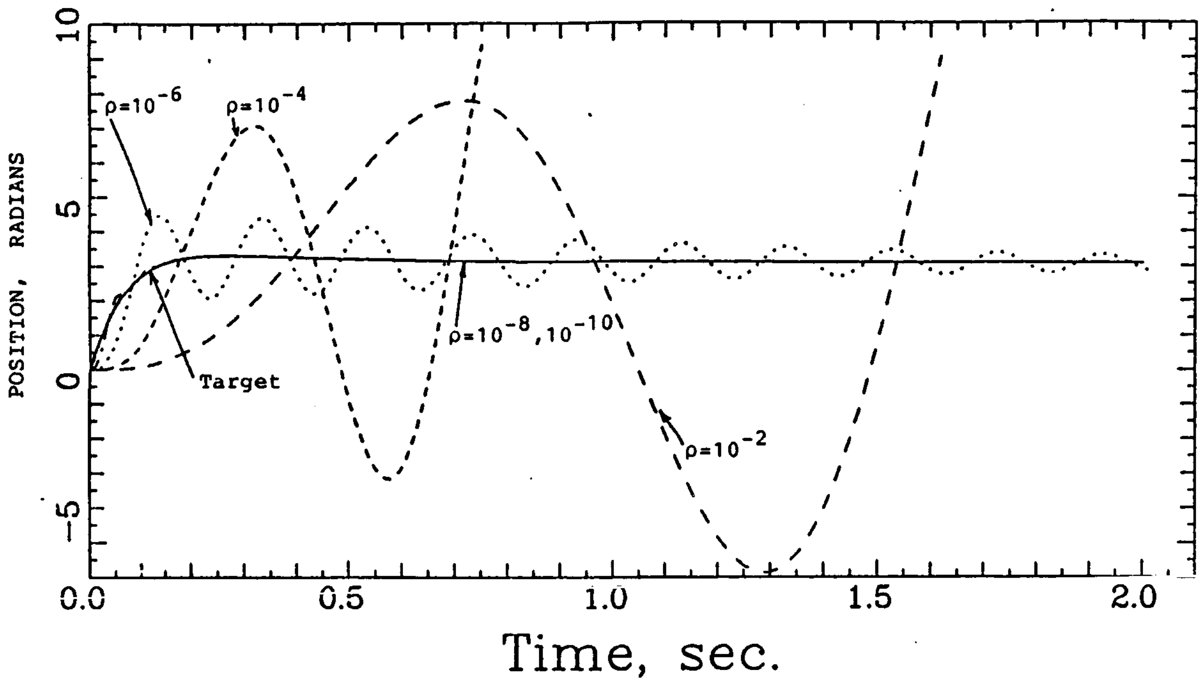


Target: $\kappa C\Phi H[I+\kappa C\Phi H]^{-1} r$

Values of ρ: $\kappa PK_{\rho}[I+\kappa PK_{\rho}]^{-1} r$

$\kappa = 2.0$

Figure 5-8: Gain Margin Check: x2

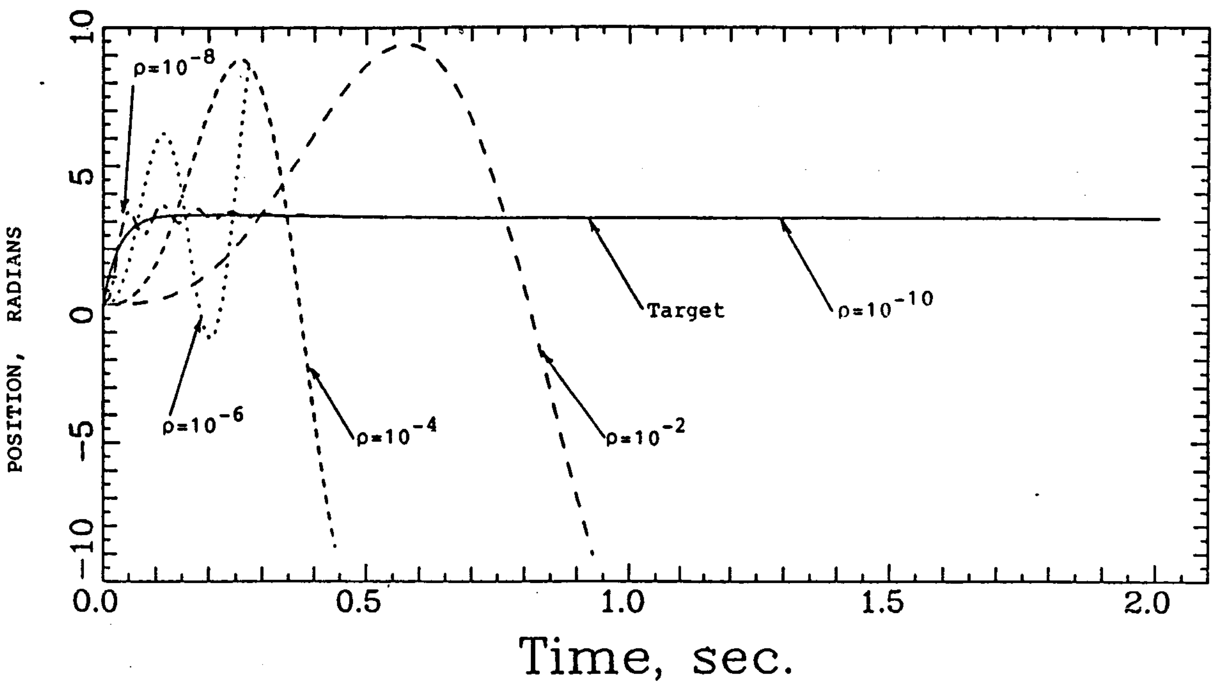


Target: $\kappa C\Phi H[I+\kappa C\Phi H]^{-1}r$

Values of ρ: $\kappa PK_{\rho}[I+\kappa PK_{\rho}]^{-1}r$

$\kappa=5.0$

Figure 5-9: Gain Margin Check: x5



Target: $\kappa C\Phi H[I+\kappa C\Phi H]^{-1}r$

Values of ρ: $\kappa PK_{\rho}[I+\kappa PK_{\rho}]^{-1}r$

$\kappa=10.0$

Figure 5-10: Gain Margin Check: x10

"frequencies" of interest, namely, up to the crossover of the perturbed system, which is higher than the original crossover, due to the action of the gain perturbation.

Thus we have shown that

(a) The target loop has a high gain margin, and

(b) The actual loops recover this gain margin as $\rho \to 0$.

5.6 Sensitivity Computation

We now turn to a demonstration of one idea for analyzing the sensitivity of nonlinear systems. Figure 5-11a shows a swept sinewave of increasing frequency, given by

$$r(t) = 3.14 \sin(t\omega(t)) \tag{5.10a}$$

$$\omega(t) = (.4) \cdot 10^{t/40} \tag{5.10b}$$

We want to use this signal to evaluate the sensitivity operator for our system and thus we must

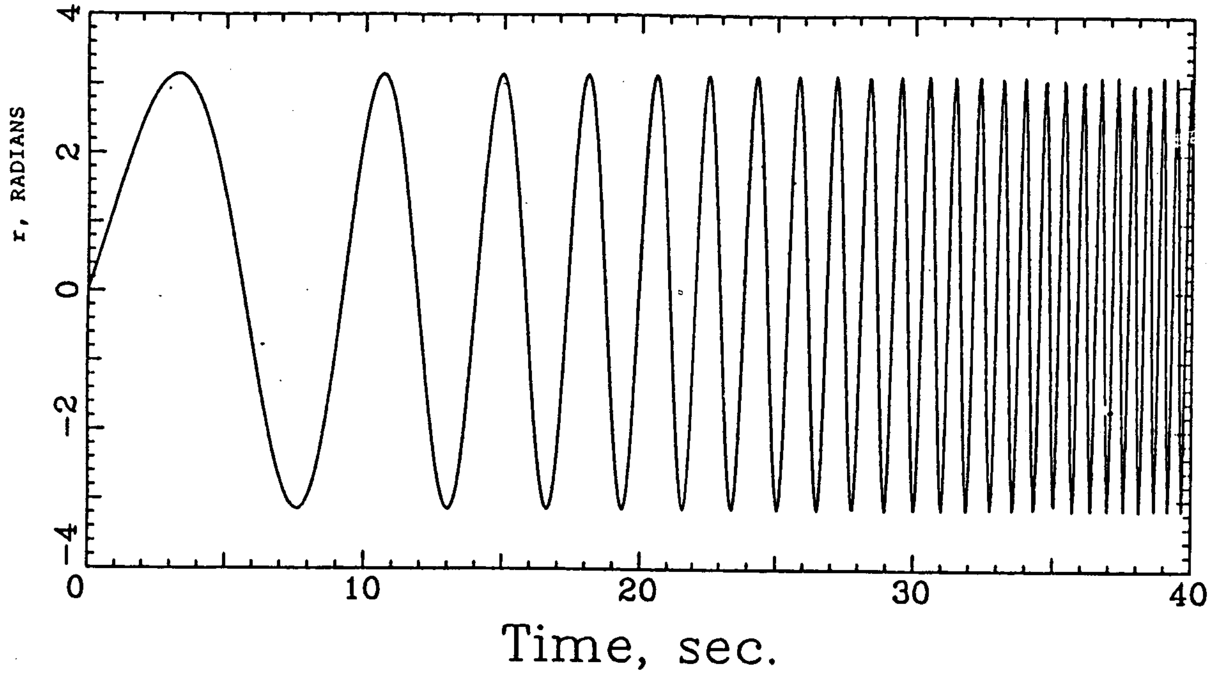
(a) choose a wide enough range to cover all the behavior of our system, and

(b) shift frequencies slowly enough that we give the system enough time to adjust to each new frequency, and

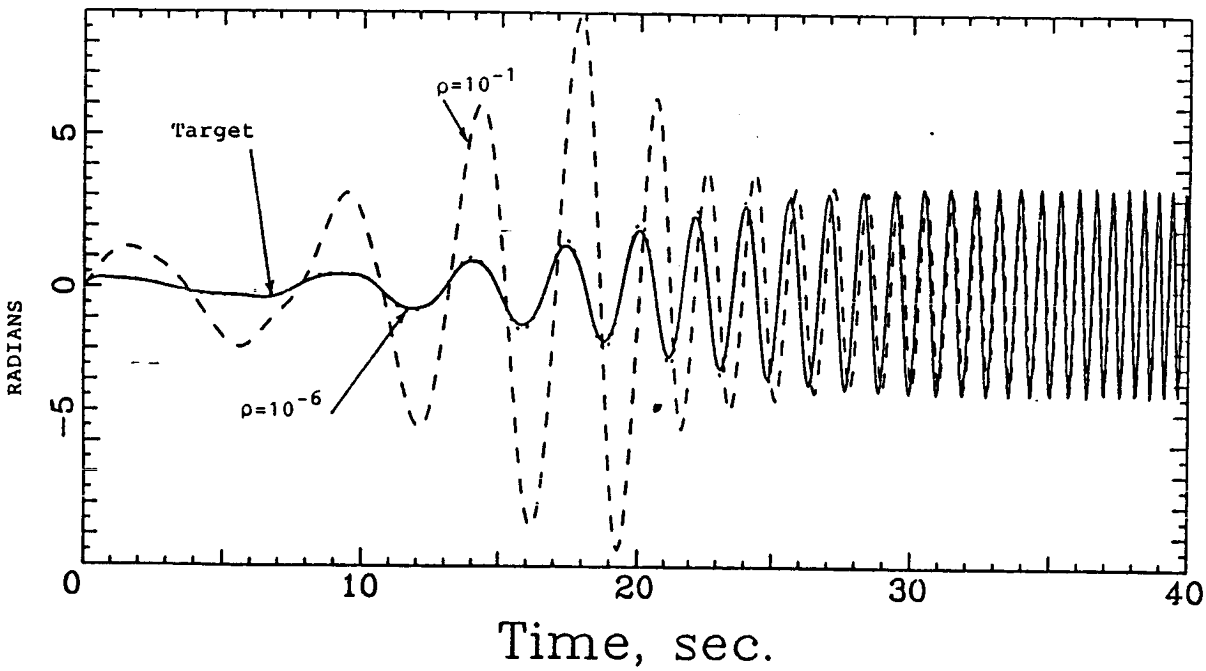
(c) use an exponential function so that the response can be viewed as having a logarithmic scale in frequency.

The r selected above (5.10) has these features.

In figure 5-11b, we show the sensitivity reponse to this signal r, i.e. plots of $[I+C\Phi H]^{-1}r$ and $[I+PK_\rho]^{-1}r$ for $\rho=10^{-1}$ and $\rho=10^{-6}$. The first thing we should observe is that recovery still takes place as we let $\rho \to 0$.



(a) Sensitivity input, r



Target: $[I+C\Phi H]^{-1}r$

Values of ρ: $[I+PK_{\rho}]^{-1}r$

(b) Sensitivities

Figure 5-11: Sensitivity Computation

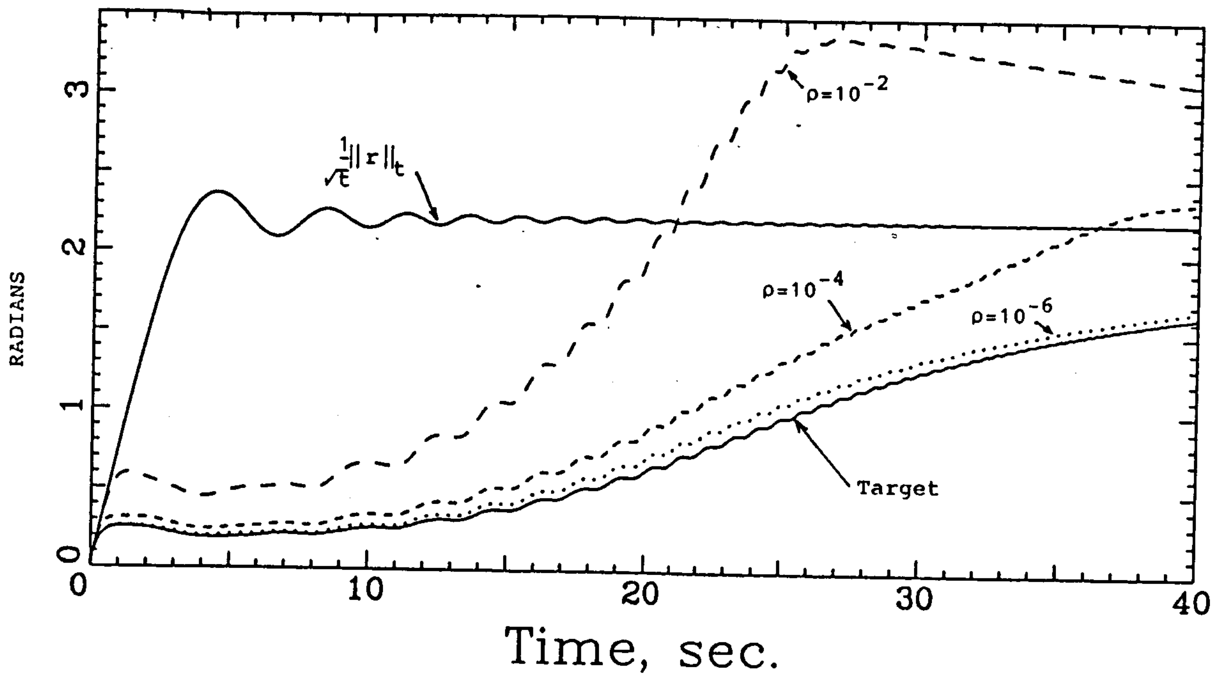
The second thing we should note is the demonstration of one of the guaranteed properties of the EKF (theorem 3.11):

$$\|[I+C\Phi H]^{-1}\| \leq 1 + \gamma \tag{5.11}$$

where γ is quite small here, since we used a fairly high gain. This is seen by the absence of "overshoot" in the sensitivity plot. If we think of the envelope of the response to r as our sensitivity plot versus frequency, we see that for all frequencies, the sensitivity operator has gain roughly less than unity. Note that by the robustness theorems of section 2.5, we know that this imples that our loop is robust.

The third, and final thing to note about figure 5-11b is that for $\rho=10^{-1}$, we do not achieve a good recovery, and in addition, the loop is not very robust, as the sensitivity operator has a gain of approximately 3 for some frequencies. Note that this peak in the sensitivity curve is not at the natural frequency of the pendulum (1 radians/second or a period of 6.28 seconds) but rather higher, at roughly 2 radians/second. This corresponds to the "crossover" frequency of the closed-loop system, with the peak occuring because of a too rapid crossover (not enough phase margin).

As another possible use of this type of sweeping sinewave is in the evaluation of L_2-norms. In figure 5-12, we plot versus time, t,



$$\frac{1}{\sqrt{t}} \|r\|_{2,t} = \left[\frac{1}{t} \int_0^t |r(\tau)|^2 \, d\tau \right]^{1/2}$$

Target: $$\frac{1}{\sqrt{t}} \|[I+C\Phi H]^{-1} r\|_{2,t} = \left[\frac{1}{t} \int_0^t |\{[I+C\Phi H]^{-1} r\}(\tau)|^2 \, d\tau \right]^{1/2}$$

Values of ρ: $$\frac{1}{\sqrt{t}} \|[I+PK_\rho]^{-1} r\|_{2,t} = \left[\frac{1}{t} \int_0^t |\{[I+PK_\rho]^{-1} r\}(\tau)|^2 \, d\tau \right]^{1/2}$$

Figure 5-12: Sensitivity Norm Computation

$$\frac{1}{\sqrt{t}}\, \|r\|_{2,t} = \left[\frac{1}{t} \int_0^t |r(\tau)|^2 \, d\tau \right]^{1/2} \tag{5.12}$$

$$\frac{1}{\sqrt{t}}\, \|[I+C\Phi H]^{-1} r\|_{2,t} \doteq \left[\frac{1}{t} \int_0^t |\{[I+C\Phi H]^{-1} r\}(\tau)|^2 \, d\tau \right]^{1/2} \tag{5.13}$$

$$\frac{1}{\sqrt{t}}\, \|[I+PK_\rho]^{-1} r\|_{2,t} = \left[\frac{1}{t} \int_0^t |\{[I+PK_\rho]^{-1} r\}(\tau)|^2 \, d\tau \right]^{1/2} , \tag{5.14}$$

for r given by (5.10). We have normalized by $\sqrt{t}$ so that the quantities do not keep growing as the simulation progresses and thus we can see everything on the same scale. Note that $(t^{-1/2})\|r\|_{2,t}$ becomes relatively constant because we are essentially plotting the average magnitude of the signal. Again, we see the recovery process, and in addition we show the inequality for the EKF loop

$$\|[I+C\Phi H]^{-1} r\|_{2,t} \leq \|r\|_{2,\tau}. \tag{5.15}$$

As we let $\rho \to 0$, we see that the actual sensitivity approaches (and meets) this inequality as well, thus showing the robustness of the recovered loop.

5.7 Conclusion

This chapter has attempted to demonstrate using a very simple example some of the NMBC/LOR techniques, in order to convince the reader that these convergence facts really are true, and that the EKF really has some guaranteed properties. Obviously, they are not the exhaustive simulations that would have to be done in an actual design, but rather, they are meant to simply illustrate some of the ideas expressed in this thesis.

CHAPTER 6. CONCLUSION

6.1 Conclusions

This thesis has proposed a new methodology for the control of multivariable nonlinear systems, that includes the issues of

(a) closed-loop stability,

(b) peformance, and

(c) stability robustness.

The methodology is an extension of a successful linear theory, LQG/LTR. Under suitable assumptions, almost all of the relevant linear theoretical results can been extended to the nonlinear case, although computationally things are more difficult.

The robustness that the results guarantee is not an "ϵ-robustness", with very small margins (e.g. there exists an $\epsilon \geq 0$ such that the gain margin is $1+\epsilon$), but rather a much better kind. Our goal is to guarantee *large* robustness margins, which translate into gain margins from -6dB to $+\infty$, and phase margins of 60 degrees, in a multivariable sense.

In the global methodology, a main feature is the unification of both I/O analysis methods and state space synthesis methods. Any practical methodology must use I/O analysis for robustness tests, as unstructured unmodeled dynamics are impossible to capture in a finite order dynamical model. Operators are relatively easy to analyze (by simulation) but extremely hard to synthesize (or calculate explicitly). On the other hand, state space methods are relatively good for calculations (the extended Kalman filter, Hamilton-Jacobi-Bellman equation, etc.). We thus split our theory in half: we do all analysis operations with I/O techniques, as discussed in chapter 2, and we do all synthesis operations with state space formulations, as discussed in chapter 3. These are then combined together to create what is hopefully a coherent methodology, as discussed in chapter 4, where the best of each ideology is utilized.

We now present a few of the less global conclusions that we wish to make.

It appears that nonlinear systems have much in common with linear time-varying systems, as is evidenced by the results of section 2.3.4. There it is shown that incremental stability of a nonlinear system is equivalent to the uniform stability of the linearized time-varying systems about all possible nominal trajectories. In addition, the extended Kalman filter for nonlinear systems (sections 3.3.3 and 3.3.4) has many guaranteed properties as a result of the properties of the linear time-varying Kalman filter (theorem 3.7).

One of the guaranteed properties of the extended Kalman filter is its nondivergence. We show in theorem 3.6 that if a system is detectable, that is, if any nondivergent estimator can be built for it, then the extended Kalman filter will also be nondivergent. This has important ramifications for controller design as well as for problems in which nonlinear estimation is the desired end result. It is important to note that even in applications where it appears on the surface that estimation is the desired end result, usually the estimated information will be fed back in some way to provide control action. In that case, a guarantee of nondivergence could be quite reassuring.

We also show in this thesis how the external linearization results of other researchers can be used in a nonlinear model-based-compensator structure in a way that does not forfeit good loop shaping properties. Transformation methods may be used to generate a model of the plant in controller form to obtain a state feedback function (section 3.3.6), which can be used in the recovery procedure at the plant output (section 4.3.3). These methods could also be used to generate a model in observer form (section 3.4.5) to obtain a nondivergent estimator.

It is hoped that the results from research on other methodologies will be able to be used with the research presented here as well. For example, the analysis tests of chapter 2 (sections 2.3, 2.4, and 2.5) are useful for judging any proposed nonlinear control methodology.

In sections 3.4.2. and 3.4.3 we presented results pertaining to optimal nonlinear regulators. Even though it is at present quite difficult to solve the Hamilton-Jacobi-Bellman (HJB) partial differential equation, we show that its solutions have some very nice guaranteed properties, and hope that this presents some motivation for further research on its computation (see the next section for some ideas).

Slightly harder to compute than the HJB equation is the costate observer partial differential equation, discussed in appendix C. The solution of this equation, has many guaranteed properties, and if certain extra conditions are satisfied, may prove to be a nondivergent estimator. It has the advantage of being a spiritual dual to the HJB equation, even more so than the EKF.

Far harder to compute than the HJB or costate equation is the optimal nonlinear filter, as discussed in section 3.3.2. This essentially requires the storage and update of every point in the state space at each discretization interval. Current computational technology is still not good enough to do this for all but the simplest problems. Perhaps further research/technology will change this. Computationally difficult though it may be to solve, the solution to the optimal filtering problem enjoys the same robustness properties of the EKF and costate observer (see appendix D).

6.2 Future Research Directions

The next three sections are devoted to discussing in some detail the possibilities for future research and extensions relating to the results presented throughout this thesis. We do this in sections roughly paralleling the chapters of the thesis: section 6.2.1 covers the analysis results, section 6.2.2 covers synthesis results, and section 6.2.3 covers the integration of the two ideas.

6.2.1 Analysis

One of the main difficulties with the analysis results of chapter 2 is the posing of specifications. As mentioned in section 2.6, we might want to use describing function ideas to pose specifications in a way that humans can comprehend (i.e. not just a listing of every possible input paired with each desired output).

A related issue is the actual calculation of tests which require that a certain condition hold for all signals in some signal space, such as is required for the robustness tests of section 2.5. We need some results that say, in effect, that if a system is smooth enough, we only need to check the conditions over some "dense" set in order to guarantee that they hold over the whole signal space. This dense set will need to be much smaller than the usual mathematical definition of a dense set; we want to be able to calculate these conditions fairly quickly.

6.2.2 Synthesis

The first thing that needs additional work in chapter 3 is the extended Kalman filter. For example, the conditions requiring controllability through $\Xi^{1/2}$ should be looked at closely so that perhaps conditions can be developed for checking this.

In addition, it would be nice to be able to relax the restriction on the boundedness of the first and second derivatives of the function f. Since the EKF is essentially a first order approximation, it appears that the iterated extended Kalman filter [39,46] might prove nondivergent under f's with some polynomial behavior with degree higher than one.

It seems also likely that the EKF should have some guaranteed stochastic properties, especially in the area of local optimality. Since no filter can be better for small noises (and thus small errors), we should be able to prove some optimal local properties. Then by the extension trick of theorem 2.4, we might be able to extend the optimality to a more global property.

Another area that might benefit from the results on the EKF is that of time-varying systems. Theorem 3.7 shows that the normal linear time-varying Kalman filter has the same robustness properties as the steady-state version. Perhaps this would be of use in the design of estimators for time-varying or gain-scheduled systems.

The Hamilton-Jacobi-Bellman partial differential equation is certainly a big research area in itself. While there has been some research on the actual calculation by polynomial approximation [56,57], other methods may have better potential. One possibility is a transformation based one. The linear version of the HJB equation is the algebraic matrix Ricatti equation, and one of the ways of solving the Ricatti equation involves factoring the Hamiltonian system into two parts: (1) a part with all the stable modes and the mirror images of the unstable modes, and (2) a part with all the unstable modes and the mirror images of the stable modes. This then allows the actual solution to be computed. It seems that a nonlinear version of this factorization may be possible, in which the

Hamiltonian system (from the Maximum Principle of Pontriagin [52,66]) can be transformed in such a way that the unstable and stable "modes" are separated, allowing one to compute the optimal cost-to-go.

Another, less exotic, method for calculating the solution to the HJB equation relies on an iterative real-time procedure. Pick any algorithm to calculate g(x) for a fixed x. This requires solving the two-point-boundary-value-problem (TPBVP) and can be done by a steepest descent algorithm, for example. The important thing to realize is that these algorithms can be designed to converge very quickly for initial trajectory guesses that are close to the optimal ones. Thus, we can use them in real-time, if our system does not move too fast. The idea is as follows. Suppose we are at time t_1. We start with an intial guess for the optimal control $u_1^*(\cdot)$ and state $x_1^*(\cdot)$ trajectories starting from the current state $x(t_1)$. We apply the control $u_1^*(t_1)$ to the system. At the next time step, we will have a slightly different state $x(t_1+\Delta t)=x(t_2)$, also different from the expected state $x_1^*(t_1+\Delta t)=x_1^*(t_2)$ due to disturbances. We then use the old trajectories as starting points in our algorithm to compute the new optimal trajectories. Since the time steps must be reasonably small, the old and new trajectories should be close, and thus we can converge quickly, perhaps in one step, to the new optimal trajectories $u_2^*(\cdot)$, $x_2^*(\cdot)$starting from $x(t_2)$. We then use as our new control $u_2^*(t_2)$ and continually repeat the process. This procedure might prove to be very computationally efficient. It also avoids the problem of storing $g(\cdot)$, i.e. g(x) for each value of x.

Another area of research involving the HJB equation relates to the incremental properties of its solution. We would like to develop conditions on the optimal control problem (i.e. on B, f, and m(•)) that would guarantee that we have an incrementally stabilizing state feedback. There has been some research on the incremental stability of optimal control solutions (called the second variation [67,68]) but only as they relate to perturbations about the nominal optimal regulator trajectory (i.e. zero input case). Perhaps these results could be extended to the case of arbitrary nominal trajectories.

If conditions could be developed for guaranteeing that the optimal regulator was incrementally stable, it seems likely that they would apply to the costate observer as well, due to the similar mathematical structure of the HJB and costate observer partial differential equations. As indicated in appendix C, if the costate observer can be made incrementally stable, it would be a nondivergent state estimator. This would provide a potentially more attractive observer than the EKF, due to its lower dimension (order n versus order n + (n+1)n/2).

6.2.3 Compensator Design

One of the important areas from chapter 4 needing additional work is the relaxation of some of the restrictions on the recovery procedures. Recovery at the plant output currently requires a quite restrictive plant model: the plant model must be in both controller and observer form (theorem 4.2). This requires that all the nonlinearities be directly controlled by the input, and be functions only of the output. One possible avenue of research would be to remove some of these restrictions on the allowable class of models for recovery at the plant output.

If these restrictions cannot be lessened, it may be possible to extend the results for the formal loop shaping (FLS) procedure (theorem 4.3). While the recovery process itself does not require an incrementally stable plant, in order to guarantee stability for the closed-loop system with FLS, we must currently have a plant model which is incrementally stable. It would certainly be desirable to remove this restriction. In the linear case, [2] shows how to do FLS with unstable plants. This indicates that there may be some possibility that the FLS compensator could stabilize an unstable plant.

Another important area for research involves the idea of minimum phase behavior. It would be helpful to have a rigorous definition for minimum phase systems that would capture the idea of non-invertibility. Perhaps the literature on invertibility of dynamical systems would be of use [69,70]. With this type of result, one could quantify the conditions under which the HJB equation gives the asymptotic behavior needed for recovery, namely that the optimal cost and its derivatives goes to zero as the control weighting goes to zero. In addition, perhaps the results of [65] could be extended to give limits on possible performance for nonlinear systems with nonminimum phase behavior.

A totally different approach to linear control systems design than that of LQG/LTR is the *factorization approach* [71], which treats systems as ratios of stable polynomials. In appendix E, we discuss how these factorization ideas might be extended to the nonlinear case. We show there how the ideas of state feedback and state estimation are related to factorization. Perhaps these ideas can be extended and made more precise.

Another idea that could use development is that of loop shaping. We tried to give a sample of this in section 4.4.2, where we used the L_2-inequality to give some rough handle on the optimal regulators loop operator, in a manner reminiscent of the properties of the Kalman frequency domain inequality [2,72]. Perhaps other asymptotic results could be derived for the optimal regulators loop operator. For example, a useful formula would be an approximation to the loop operator "at high frequencies", i.e. for those signals where $G\Phi B$ is small. This would allow a designer to control the aprroximate crossover behavior of the closed-loop system, as is done in the LQG/LTR methodology.

Along the lines of loop shaping, we remark that in the linear theory there has been a lot of research on H_∞ control design methods, where the H_∞-norm of weighted sensitivity functions is minimized. In [2] it is shown that LQG/LTR can be interpreted in terms of an H_2 minimization of weighted sensitivities. We now show here how we might interpret a minimization of the system closed-loop gain as a minimization of weighted sensitivities, as was done for the linear case [2]. Let

$$\dot{x} = f(x) + Bu + Bw \tag{6.1a}$$

$$y = Cx + \mu d \tag{6.1b}$$

$$z = Mx, \tag{6.1c}$$

where M may be a nonlinear function of x, and the variable z is an auxiliary variable chosen as the important variable that will be weighted in the optimization problem. The function M is analogous to the state weighting function of the optimal regulator theory, m(x), i.e.

$$|m(x(t))| = |z(t)|^2 = |Mx(t)|^2. \tag{6.2}$$

Let K be a given compensator, and let $Q_K:(w,d)\mapsto(z,u)$ as depicted in the closed-loop arrangement of figure 6-1. Suppose that we can solve the following minimization problem:

L_p-Minimization Find K to minimize

$$\|Q_K\|_p \quad := \quad \sup_{\substack{w,d \\ \tau>0}} \frac{\|Q_K(w,d)\|_{p,\tau}}{\|(w,d)\|_{p,\tau}}. \tag{6.3}$$

We now pose a different minimization problem, which we will show is related to $\mathcal{L}_p$-minimization.

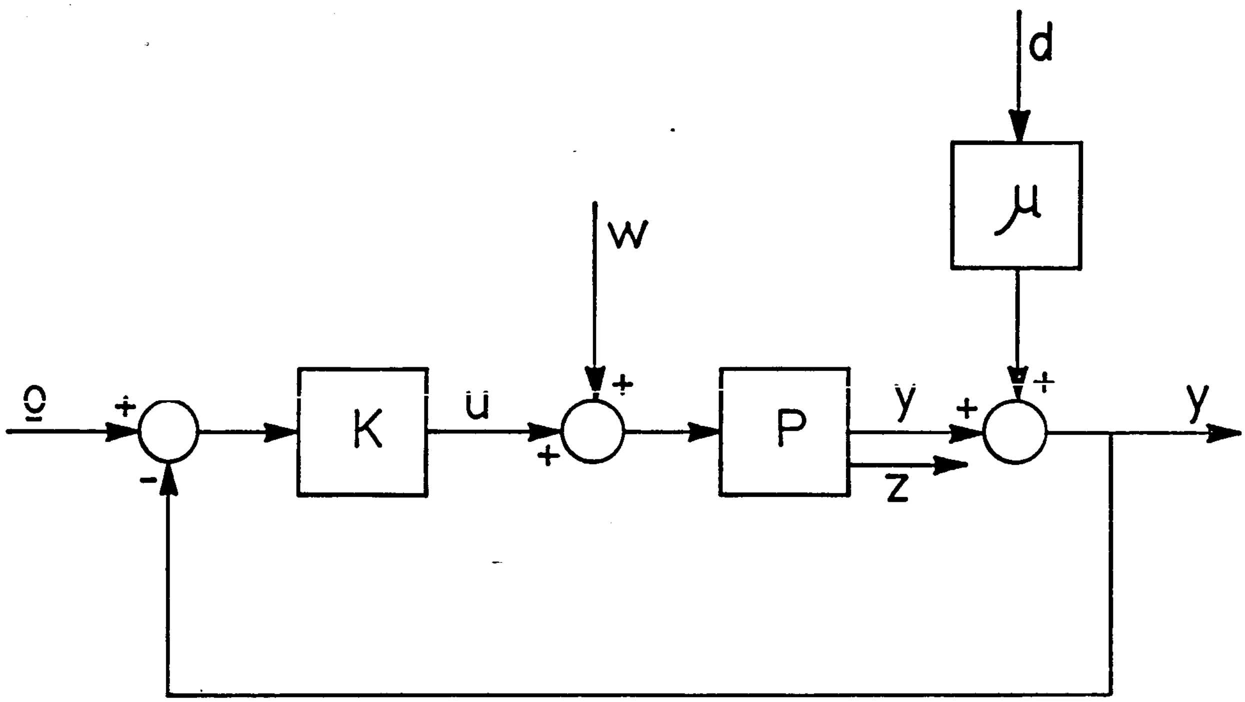


Figure 6-1: L_p-Minimization Problem

Weighted L_p-Sensitivity Minimization Find K to minimize:

$$\sup_{\substack{w\\ \tau>0}} \frac{\|Q_K(w,0)\|_{p,\tau}}{\|(w,0)\|_{p,\tau}} = \sup_{\substack{w\\ \tau>0}} \left[\frac{\|JSw\|^2_{p,\tau} + \|Tw\|^2_{p,\tau}}{\|w\|^2_{p,\tau}}\right]^{1/2} \tag{6.4}$$

where J is a given weighting operator, and

$$S = [I+(-K)(-P)]^{-1} \tag{6.5}$$

is the sensitivity and

$$T = I - S \tag{6.6}$$

is the complementary sensitivity.

This problem for p=2 is similar to the H_∞ problem of linear systems theory. It is quite easily proven that if we let $J=M\Phi B$ and let $\mu\to0$, then the solution to the L_p-minimization problem becomes the solution to the corresponding weighted L_p-sensitivity minimization problem.

This raises some interesting points:

(a) Suppose that we could solve the L_p-minimization problem (6.3) (which may be quite difficult). Then we could solve the weighted L_p-sensitivity minimization problem (6.4), using the same algorithm.

(b) How can the L_p-minimization problem be solved? Perhaps it involves the solution of the general stochastic optimal output feedback control problem [53]; hopefully it will be much easier.

(c) To what extent does the NMBC/LOR or FLS compensator presented in this thesis approximate the solution to either of the above minimization problems?

(d) How might this type of analysis be of help in extending the H_∞ design methods to the nonlinear case, or perhaps in helping us to understand the NMBC/LOR methodology more fully?

C-3

APPENDIX A. Proof of Theorem 2.4

Proof (if) The derivative condition on f implies [8] that given an $\epsilon>0$ there exists a $\delta_m(\epsilon)$ so that

$$|\delta| \leq \delta_m(\epsilon) \implies |g(x,\delta)| \leq \epsilon\, |\delta| \tag{A.1}$$

where

$$g(x,\delta) := f(x+\delta,t) - f(x,t) - \nabla f(x,t)\delta. \tag{A.2}$$

Let $\Phi(t,\tau)$ be the state transition matrix for the linear time-varying system (2.52). Since it is L_∞ stable, there exists an N such that [8]

$$\int_0^\infty |\Phi(t,\tau)|\, d\tau \leq N \tag{A.3}$$

Consider two trajectories of (2.50):

$$x_1 = Pu \tag{A.4}$$

$$x_2 = P(u+w) \tag{A.5}$$

for any u,w and let $e=x_2-x_1$ be the error between them. We now make use of the following intermediate result.

Lemma A.1 [8]   For the above situation, we have

(a) For all $\epsilon>0$, $\|e\|_{\infty,T} \le \delta_m(\epsilon)$ implies that

$$\|e\|_{\infty,T} \le \frac{1}{1-\epsilon N} \|\xi\|_{\infty,T} \le \frac{k}{1-\epsilon N} \|w\|_{\infty,T} \tag{A.6}$$

(b) If $\|w\|_{\infty,T} \le \frac{1-\epsilon N}{k} \delta_m$, then (A.6) holds.

Proof of Lemma   We have

$$\dot{e} = f(x_2,t)-f(x_1,t) = \nabla_x f(x_1,t)e + g(x_1,e) + w \tag{A.7}$$

and so

$$\begin{aligned} e(t) &= \int_0^t \Phi(t,\tau)[w(\tau) + g(x_1(\tau),e(\tau))]d\tau \\ &= \xi(t) + \int_0^t \Phi(t,\tau)g(x_1(\tau),e(\tau))d\tau \end{aligned} \tag{A.8}$$

Pick $\epsilon<1/N$. If $\|e\|_\infty \le \delta_m(\epsilon)$ then

$$\begin{aligned} |e(t)| &\le |\xi(t)| + \int_0^t \epsilon|\Phi(t,\tau)|\cdot|e(\tau)|d\tau \\ &\le |\xi(t)| + \epsilon N \|e\|_{\infty,t} \end{aligned} \tag{A.9}$$

and

$$\|e\|_{\infty,T} \le \|\xi\|_{\infty,T} + \epsilon N \|e\|_{\infty,T} \tag{A.10}$$

Since we picked ϵ so that $\epsilon N<1$, we have result (a) of the Lemma. Result (b) follows from (a), since as we pick w small enough, ξ is bounded and thus e is small enough to guarantee the hypothesis of (a). ∎ (Lemma)

We now finish the proof of the theorem by extending the lemma result to the entire input space, not just small inputs. Let $w \in \mathcal{L}$ and $\tau \in \mathbb{R}_+$ be arbitrary. Let $r := \|w\|_\tau$ which must be finite. Now pick an integer n large enough so that

$$r < n \frac{1-\epsilon N}{k} \delta_m \tag{A.11}$$

Then we have

$$\begin{aligned}
\|e\|_{\infty,\tau} &= \|Pu - P(u+w)\|_{\infty,\tau} \\
&= \Big\| Pu - P(u+w\tfrac{1}{n}) + P(u+w\tfrac{1}{n}) - P(u+w\tfrac{2}{n}) + P(u+w\tfrac{2}{n}) - \ldots \\
&\qquad - P(u+w\tfrac{n-1}{n}) + P(u+w\tfrac{n-1}{n}) - P(u+w\tfrac{n}{n}) \Big\|_{\infty,\tau} \\
&\leq \Big\| Pu - P(u+w\tfrac{1}{n}) \Big\|_{\infty,\tau} + \Big\| P(u+w\tfrac{1}{n}) - P(u+w\tfrac{2}{n}) \Big\|_{\infty,\tau} + \ldots \\
&\qquad + \Big\| P(u+w\tfrac{n-1}{n}) - P(u+w\tfrac{n}{n}) \Big\|_{\infty,\tau} \\
&\leq n \cdot \frac{1}{n} \cdot \frac{k}{1-\epsilon N} \|w\|_{\infty,\tau} = \frac{k}{1-\epsilon N} \|w\|_{\infty,\tau},
\end{aligned} \tag{A.12}$$

because

$$\|\tfrac{1}{n} w\|_{\infty,\tau} = \frac{r}{n} \leq \frac{1-\epsilon N}{k} \delta_m \tag{A.13}$$

is small enough to allow us to use the Lemma.

Proof (only if) Select an arbitrary trajectory $x_1 \in \mathcal{L}$ with associated input u as in (A.8). Select a trajectory pair $v, \xi \in \mathcal{L}$ for (A.3), fix $\tau \in \mathbb{R}_+$, and pick $\epsilon < 1/k$. Now let

$$\gamma = \frac{\delta_m(\epsilon)}{\|\xi\|_{\infty,\tau}} . \tag{A.14}$$

If $w \in \mathcal{L}$ is chosen so that $e = \gamma\xi$ then

$$\gamma v = g(x_1, e) + w \tag{A.15}$$

and

$$\|\xi\|_{\infty,\tau} = \|\frac{1}{\gamma} \cdot e\|_{\infty,\tau} \le \frac{k}{\gamma} \|w\|_{\infty,\tau} = \frac{k}{\gamma} \|\gamma v - g(x_1, e)\|_{\infty,\tau}$$

$$\le k \|v\|_{\infty,\tau} + \epsilon k \|\xi\|_{\infty,\tau} . \tag{A.16}$$

Therefore

$$\|\xi\|_{\infty,\tau} \le \frac{k}{1-\epsilon k} \|v\|_{\infty,\tau} \tag{A.17}$$

and since τ was arbitrary, and ϵ could have been any smaller number, we have the desired result. ∎

APPENDIX B. Proof of Theorem 3.6

(c) implies EKF nondivergent For the linearized EKF system

$$\dot{\xi} = \nabla f(\hat{x}(t))\xi + Bu + H(t)[y-C\xi], \tag{B.1}$$

let

$$v(\xi,t) = \frac{1}{2}\,\xi^T\Sigma^{-1}(t)\xi. \tag{B.2}$$

Then (B.2) implies

$$\frac{1}{2\beta}\,|\xi|^2 \le v(\xi,t) \le \frac{1}{2\alpha}\,|\xi|^2 \tag{B.3}$$

and along trajectories of (B.1) with zero input (u,y=0)

$$\frac{dv(\xi,t)}{dt} = -\frac{1}{2}\,\xi^T\Sigma^{-1}(t)\dot{\Sigma}(t)\Sigma^{-1}(t)\xi + \xi^T\Sigma^{-1}(t)\dot{\xi}$$

$$= -\frac{1}{2}\,\xi^T\Sigma^{-1}(t)\dot{\Sigma}(t)\Sigma^{-1}(t)\xi + \xi^T\Sigma^{-1}(t)\{\ \nabla f(\hat{x}(t))\xi - H(t)C\xi\ \}$$

$$=\frac{1}{2}\xi^T\Sigma^{-1}(t)\Big\{-\dot{\Sigma}(t) + \nabla f(\hat{x}(t))\Sigma(t) + \Sigma(t)\nabla f(\hat{x}(t)) - 2\Sigma(t)C^TC\Sigma(t)\Big\}\Sigma^{-1}(t)\xi$$

$$= -\frac{1}{2}\xi\Sigma^{-1}(t)\Big\{\ \Xi + \Sigma(t)C^TC\Sigma(t)\Big\}\Sigma^{-1}(t)\xi$$

$$\le -\frac{1}{2}\,\epsilon\,\left[\frac{1}{\beta}\right]^2 |\xi|^2. \tag{B.4}$$

Also, we have

$$\left|\frac{\partial v(\xi,t)}{\partial x}\right| \leq \sigma_{max}[\Sigma^{-1}(t)] \; |\xi| \leq \frac{1}{\alpha} \; |\xi|. \tag{B.5}$$

Since (B.3-B.5) hold for all $\xi \in \mathbb{R}^n$, we can apply theorem 2.2 to conclude that (B.1) is exponentially stable, i.e. there existconstants $\lambda, M > 0$ such that trajectories of (B.1) obey

$$|\xi(t)| \leq M \; |\xi_o| \; e^{-\lambda t} \tag{B.6}$$

for all initial conditions $\xi(0)=\xi_o$, with $u,y=0$.

Thus, by theorem 2.3, (B.1) is uniformly L_2 and L_∞-stable (with $\xi_o=0$) for all matrices B and all trajectories $\hat{x}$. Now, we would like to apply theorem 2.4 to conclude that the EKF is nondivergent, however, since the EKF has a slightly different form than theorem 2.4 used, due to the dependency of H on Σ, we must prove it directly here. Refering to theorem 2.4 for guidance will help.

We start with (B.1) being uniformly I/O stable, and

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = f(x) - f(\hat{x}) - H(t)Ce + Bw - H(t)d \\ &= \nabla f(\hat{x}(t))e + g(\hat{x},e) - H(t)Ce - H(t)d + Bw \end{aligned} \tag{B.7}$$

and letting Φ be the state transition matrix for (B.1),

$$e(t) = \int_0^t \Phi(t,\tau)[\ - H(t)d + Bw + g(\hat{x},e)\ ]\ d\tau$$

$$= \xi(t) + \int_0^t \Phi(t,\tau)g(\hat{x}(\tau),e(\tau))d\tau. \tag{B.8}$$

We can now finish the proof in the manner of theorem 2.4 to conclude

$$\|e\|_\tau \leq k\ \|(w,d)\|_\tau;\ \forall\tau\in\mathbb{R}_+,w,d\in\mathcal{L}, \tag{B.9}$$

which is the desired conclusion.

(b) implies (c) We use the following result of Bucy&Joseph [49, chapter V] for linear time varying systems.

Lemma B.1 For the time-varying linear system $[A(\cdot),B(\cdot),C(\cdot)]$ and the associated Kalman filter

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + \Xi - \Sigma(t)C(t)^TC(t)\Sigma(t), \tag{B.10}$$

(a) if $[A(\cdot),C(\cdot)]$ is uniformly observable, then for all $t>t_0+\sigma$, where σ is the interval of observability, and for all Σ_0

$$\Sigma(t) \leq [W^{-1}(t,t-\sigma) + C(t,t-\sigma)]. \tag{B.11}$$

(b) if $[A(\cdot),\Xi^{1/2}]$ is uniformly controllable, then for all $t>t_0+\sigma$, where σ is the interval of observability, and for all Σ_0,

$$[C^{-1}(t,t-\sigma) + W(t,t-\sigma)]^{-1} \leq \Sigma(t) \tag{B.12}$$

Proof see [49]. ∎

Now, since W and C are uniformly bounded by hypothesis across all time-varying systems (i.e. for all $\hat{x}$) we obtain uniform bounds on $\Sigma(t)$, and thus by (c), the EKF is nondivergent for $t_0<-\sigma$.

(a) implies (c) This is the hardest proof of the theorem; it is also the most significant result. We proceed by a series of lemmas. Readers not interested in the details can scan the lemmas for a sketch of the proof.

Lemma B.2 For all admissable trajectories $z(\cdot)\in\mathcal{L}$ that can be achieved by

$$\dot{z}(t) = f(z(t)) + u(t); \quad z(0)=0, \tag{B.13}$$

where [f,C] is M-detectable, there exists a time-varying matrix $H^*(t)$ that makes

$$\dot{\xi}(t) = [\nabla f(\hat{x}(t)) - H^*(t)C]\xi(t) + v(t) \tag{B.14}$$

L_∞-stable, uniformly for all $z(\cdot)$, i.e. there exists $k>0$ such that

$$\|\xi\|_{\infty,\tau} \leq k \; \|v\|_{\infty,\tau} \tag{B.15}$$

for all v,ξ satisfying (B.14) and for all $\tau \in \mathbb{R}_+$.

Proof Since the system [f,C] is M-detectable, there must exist a nondivergent estimator with associated functional $H(\cdot,\cdot,\cdot)$ and continuity function $\eta(\epsilon,\tau)$. Since, by definition, this estimator must be nondivergent for all B matrices in the plant and the estimator, with uniform gain k, we can select B=I. The estimator is given by

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + u(t) + H(t,\ y(s),u(s),0\leq s<t)[y(t)-C\hat{x}(t)];$$
$$\hat{x}(0)=0. \tag{B.16}$$

For this proof, set d=0. Select an admissable pair u,z satisfying (B.13) and define

$$\dot{x}(t) = f(x(t)) + u(t) + w(t);\ x(0)=0 \tag{B.17a}$$
$$y(t) = C\,x(t), \tag{B.17b}$$

Let

$$g(\hat{x}(t),e(t)) \;:=\; f(x(t)) - f(\hat{x}(t)) - \nabla f(\hat{x}(t)) \tag{B.18}$$

in a manner similar to (A.2) of appendix A, where $e = x - \hat{x}$ is the estimation error. The estimation error obeys

$$\dot{e}(t) = [\nabla f(\hat{x}(t)) - H(t, y(s), u(s), 0 \leq s < t)]Ce(t) + g(\hat{x}(t), e(t)) + w(t) \tag{B.19}$$

Fix $\tau \in \mathbb{R}_+$ and pick an arbitrary trajectory pair v, ξ for the linearized system

$$\dot{\xi}(t) = [\nabla f(\hat{x}(t)) - H(t, y(s), u(s), 0 \leq s < t)]\xi(t) + v(t). \tag{B.20}$$

We now compute the gain for the linearized system (B.20). Pick

$$\epsilon < \frac{1}{k} \tag{B.21}$$

and let

$$\gamma = \frac{\delta_m(\epsilon)}{\|\xi\|_{\infty, \tau}} \tag{B.22}$$

where $\delta_m(\epsilon)$ is the continuity function for $g(\cdot,\cdot)$ from (A.1). We now select w so that

$$e(t) = \gamma\, \xi(t). \tag{B.23}$$

The w we will need is thus determined by comparing (B.19) and (B.20) and setting

$$\gamma v(t) = g(\hat{x}(t), e(t)) + w(t). \tag{B.24}$$

Since

$$\|e\|_{\infty,T} \leq \|\gamma\xi\|_{\infty,T} \leq \delta_m(\epsilon) \tag{B.25}$$

we have

$$\|\xi\|_{\infty,T} = \|\frac{1}{\gamma}e\|_{\infty,T} \leq \frac{k}{\gamma}\|w\|_{\infty,T} \leq \frac{k}{\gamma}[\|\gamma v-g(\hat{x},e)\|_{\infty,T}]$$

$$\leq k\|v\|_{\infty,T} + \frac{k}{\gamma}\epsilon\|e\|_{\infty,T},$$

$$\leq k\|v\|_{\infty,T} + k\epsilon\|\xi\|_{\infty,T}. \tag{B.26}$$

Therefore

$$\|\xi\|_{\infty,T} \leq \frac{1}{1-\epsilon k}\|v\|_{\infty,T}. \tag{B.27}$$

We now make use of the continuity of solutions of differential equations with respect to parameter variations [79, p.29] to obtain the desired final result. Let

$$H^*(t) \quad := \quad H(t,Cz(s),u(s),0\leq s<t). \tag{B.28}$$

As we let $\epsilon \to 0$, we have pointwise in time, $w \to 0$, and thus

$$x \to z \tag{B.29}$$

$$y=Cx \to Cz \tag{B.30}$$

$$H(t,y(s),u(s),0\leq s<t) \to H^*(t) \tag{B.31}$$

$$\hat{x} \to x \tag{B.32}$$

$$\nabla f(\hat{x}(t)) \to \nabla f(z(t)) \tag{B.33}$$

with solutions of (B.20) satisfying (B.27) for all $\epsilon>0$. Therefore, solutions of the limit equation (B.14) must obey (B.15) for v and for all $\tau\in\mathbb{R}_+$. Since the $z(\cdot)$ we originally picked was arbitrary, we are done. ∎ (Lemma B.2)

Lemma B.3 The time-varying system (B.14) is uniformly controllable, with arbitrary interval of controllability, σ, uniform across all trajectories z.

Proof Let

$$A_F(t) = \nabla f(\hat{x}(t)) - H^*(t)C \tag{B.34}$$

$$|A_F(t)| \leq N \tag{B.35}$$

where N exists by the bounds on ∇f and H^*. Select a $x_1=\mathbb{R}^n$, with $|x_1|=1$ and let x be the trajectory from 0 to x_1 over σ units of time:

$$x(t) = x_1 t/\sigma \tag{B.36}$$

and v(t) must be

$$\dot{x}(t) = x_1/\sigma = A_F(t)x(t) + v(t) \tag{B.37}$$

$$v(t) = [I - A_F(t)t]\ x_1/\sigma, \tag{B.38}$$

$$|v(t)| \leq (1+N)|x_1| \tag{B.39}$$

Now, we also have that

$$x_1^T x_1 = x_1^T x(t_o+\sigma) = \int_{t_o}^{t_o+\sigma} x_1^T \Phi(t_o+\sigma,\tau)v(\tau)d\tau, \tag{B.40}$$

and by the Schwartz inequality

$$x_1^T x_1 \le \left[\int_{t_o}^{t_o+\sigma} |x_1^T \Phi(t_o+\sigma,\tau)|^2 dt \right]^{1/2} \cdot \left[\int_{t_o}^{t_o+\sigma} v(\tau) d\tau \right]^{1/2} \quad \text{(B.41)}$$

or, using the controllability grammian, C, we have

$$1 \le x_1^T C(t_o, t_o+\sigma) x_1 \cdot (1+N) \quad \text{(B.42)}$$

and thus

$$C(t_o, t_o+\sigma) \ge \frac{1}{1+N} \quad \text{(B.43)}$$

and since N is independent of t_o, σ, and z, we conclude that the system (B.14) is uniformly controllable. ∎ (lemma B.3)

Lemma B.4 A uniformly controllable time-varying system

$$\dot{\xi}(t) = A(t)\, \xi(t) + B(t)\, u(t) \quad \text{(B.44)}$$

is L_∞-stable if and only if it is exponentially stable, i.e. there exist λ, M such that

$$|\xi(t)| \le M|\xi_o| e^{-\lambda(t-t_o)}; \; \xi(t_o)=\xi_o, \; v=0. \quad \text{(B.45)}$$

and

$$|\Phi(t,t_o)| \le M\, e^{-\lambda(t-t_o)} \; , \quad \text{(B.46)}$$

where Φ is the state transition matrix for (B.44). Furthermore, if the output is considered to be $y=C\xi$, the system will be exponentially stable if the additional constraint of uniform observability is imposed.

Proof See [73]. For related material, see [74] for the linear case, and [12] for a treatment of the general nonlinear case. ∎ (lemma B.4)

Lemma B.5 If $A(t)-H^{*}(t)C$ is exponentially stable, the covariance propagation equation for the linear filter

$$\dot{\hat{\xi}}(t) = A(t)\hat{\xi}(t) + u(t) + H^{*}(t)[y(t)-C\hat{\xi}(t)] \tag{B.47}$$

driven by white noise with intensity Ξ, with unit intensity observation noise, is bounded as follows.

$$\dot{S}(t) = [A(t)-H(t)C]S(t) + S(t)[A(t)-H(t)C]^{T} + \Xi + H(t)H^{T}(t). \tag{B.48}$$

implies

$$|S(t)| \le [\, S_o + |\Xi|\,\frac{1}{2\lambda}\,]\, N^{2};\quad t \ge t_o, \tag{B.49}$$

and

$$|S(t)| \le [\, 1 + |\Xi|\,\frac{1}{2\lambda}\,]\, N^{2};\quad t \ge t_o + \frac{\max(0, \ln\,|S_o|)}{2\lambda}. \tag{B.50}$$

where λ, N are the constants of the exponential stability.

Proof From standard linear theory [50]:

$$S(t) = \Phi(t,t_o)S_o\Phi^T(t,t_o) + \int_{t_o}^{t} \Phi(t,\tau)\Xi\Phi^T(t,\tau)\, d\tau, \tag{B.51}$$

and we have

$$\begin{aligned} |S(t)| &\le S_oN^2 \cdot e^{-2\lambda(t-t_o)} + |\Xi| \cdot N^2 \cdot \int_{t_o}^{t} e^{-2\lambda(t-\tau)} d\tau \\ &\le S_oN^2 \cdot e^{-2\lambda(t-t_o)} + |\Xi| \cdot N^2 \frac{1}{2\lambda} [\, 1 - e^{-2\lambda(t-t_o)}] \\ &\le S_oN^2 \cdot e^{-2\lambda(t-t_o)} + |\Xi| \cdot N^2 \frac{1}{2\lambda}. \end{aligned} \tag{B.52}$$

From this we easily obtain the desired bounds. ∎ (lemma B.5)

Lemma B.6 The Kalman filter for the time-varying system in the last lemma has a lower covariance than that given by (B.48).

Proof This is trivial as the Kalman Filter has the lowest covariance at any time $t \ge t_o$ of any filter [50,75].
For a intuitive explanation, we have from (B.48)

$$\dot{S}(t)=A(t)S(t)+S(t)A^T(t)+\Xi+[H^*(t)-S(t)C^T][H^*(t)-S(t)C^T]^T-S(t)C^TCS(t). \tag{B.53}$$

The Kalman filter equation is

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + \Xi - \Sigma(t)C^TC\Sigma(t), \tag{B.54}$$

and by comparing them, it is easy to see that

$$\Sigma(t) \leq S(t); \quad \forall t>t_o, \quad \Sigma(t_o)=S(t_o). \tag{B.55}$$

█ (lemma B.6)

Lemma B.7 $\Sigma(t)$ in the EKF is uniformly bounded from above for $t \geq t_o+\sigma$, where σ depends on the initial condition $\Sigma(t_o)=\Sigma_o$. This is independent of the noises, controls, etc.

Proof From the last lemma, $\Sigma(t)$ is bounded by $S(t)$, which is bounded from above. Since the bounds on $S(t)$ are uniform for all trajectories $\hat{x}$, and all u,w, and d, we have the desired result.

█ (lemma B.7)

Lemma B.8 $\Sigma(t)$ in the EKF is bounded from below for $t \geq t_o+\sigma$ if the system is uniformly controllable.

Proof From the lemma B.1, we have

$$[C^{-1}(t_o,t_o+\sigma) + W(t_o,t_o+\sigma)]^{-1} \leq \Sigma(t); \quad t \geq t_o+\sigma. \tag{B.56}$$

As mentioned previously, W has an upper bound because $A(t)=\nabla f(\hat{x}(t))$ is bounded. We shall compute that bound. Let

$$\dot{\xi}(t) = A(t)\xi(t); \quad \xi(t_o)=\xi_o \tag{B.57}$$

or

$$\xi(t) = \xi_o + \int_{t_o}^{t} A(\tau)\xi(\tau)d\tau. \tag{B.58}$$

Using the Bellman-Gronwall Inequality [8], we get

$$|\xi(t)| \le |\xi_o| \exp\{\int_{t_o}^{t} A(\tau)d\tau\}, \quad \forall t \ge t_o$$

$$\le |\xi_o| e^{-M(t-t_o)}, \tag{B.59}$$

where

$$|A(t)| = |\nabla f(\hat{x}(t))| \le M. \tag{B.60}$$

Therefore

$$\Phi(t, t_o) \le e^{-M(t-t_o)}, \tag{B.61}$$

and

$$W(t_o, t_o+\sigma) = \int_{t_o}^{t_o+\sigma} \Phi^T(\tau, t_o+\sigma)C^T C\Phi(\tau, t_o+\sigma)d\tau$$

$$\le |C|^2 \int_{t_o}^{t_o+\sigma} e^{-2M(t_o+\sigma-\tau)} d\tau$$

$$\le \frac{1}{2M} [1 - e^{-2M\sigma}]$$

$$\le \frac{1}{2M}. \tag{B.62}$$

Therefore,

$$\sigma_{min}[C^{-1}+W]^{-1} = \frac{1}{\sigma_{max}[C^{-1}+W]} \geq \frac{1}{|C^{-1}| + |W|} \geq \frac{1}{|C^{-1}| + \frac{1}{2M}}$$

$$= \frac{1}{\frac{1}{\sigma_{min}[C]} + \frac{1}{2M}} \geq \frac{2M}{2\alpha M + 1}, \qquad \text{(B.63)}$$

where α is the constant of uniform controllability. Thus $\Sigma(t)$ is bounded from below for $t \geq t_o + \sigma$, by (B.61) and (B.62). ∎ (lemma B.8)

Lemma B.9 We now finally conclude that the EKF is nondivergent.

Proof $\Sigma(t)$ is bounded from above and below, and we can use (c) of the theorem. ∎ (lemma B.9)

∎ (End of Proof of theorem 3.6)

APPENDIX C. The Costate Observer

In this appendix we will discuss a possibility for nonlinear estimation which appears to be new. We call it the *costate observer*. Because it has not been fully developed, we can only give an overview of the observer, with some of its guaranteed properties.

We first present the defining equations for the costate observer for the nonlinear system

$$\dot{x}(t) = f(x(t)) + B\,u(t) \tag{C.1}$$

$$y(t) = C\,x(t). \tag{C.2}$$

Let $S:\mathbb{R}^n \to \mathbb{R}_+$, with $S_x = [\partial S/\partial x]^T$, and S_{xx} the matrix of second partials, satisfy the partial differential equation:

$$0 = \frac{1}{2}\,p^T C^T C p - p^T S_{xx}(p) f(p) - \frac{1}{2}\,S_x{}^T(p)\Xi S_x(p). \tag{C.3}$$

The state estimate is given by

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + B\,u(t) + H(\hat{x})[y(t) - C\hat{x}(t)] \tag{C.4}$$

where

$$H(\hat{x}) := S_x{}^{-1}(\hat{x})C^T. \tag{C.5}$$

We see that the estimate equation (C.4) is a model-based estimate, as we would expect, similar to the optimal filter, the EKF, and the

CGEKF. The equation for the costate observer (C.3) comes from the HJB equation via a state variable transformation, where we let $S_x(p)=x$ and $V_x(x)=p$, i.e. so that $V_x(S_x(p))=p\ \forall p$. If we then transform the closed-loop state equations, we see that we need to make a simple modification to obtain what resembles observer dynamics. While this is the original motivation for the costate observer, it seems to stand on its own, with some properties that can be shown without reference to the HJB equation:

Theorem C.1 (Guaranteed Properties of the Costate Observer) Assume that a solution to (C.3) exists. Let H be the operator defined by $Hy := H(\hat{x})y$, in a similar manner to the way we handled the EKF time-varying gain. Then the following hold with respect to the filter loop, as shown in figure 3-2 with H(t) replaced with H with $C\Phi H: u \mapsto y$ defined by

$$\dot{x} = f(x) + H(x)u \tag{C.6a}$$

$$y = C\,x \tag{C.6b}$$

(a) Return Difference Condition

$$\frac{\|[I+C\Phi H]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq 1; \qquad \forall u \in \mathcal{L},\ u \neq 0,\ \tau \in \mathbb{R}_+. \tag{C.7}$$

(b) Other Robustness Properties

$$\frac{\|C\Phi H[I+C\Phi H]^{-1}u\|_{2,\tau}}{\|u\|_{2,\tau}} \leq 2; \qquad \forall u\in\mathcal{L},\ u\neq 0,\ \tau\in\mathbb{R}_+. \tag{C.8}$$

$$\frac{\|[I+(C\Phi H)^{-1}]u\|_{2,\tau}}{\|u\|_{2,\tau}} \geq \frac{1}{2}; \qquad \forall u\in\mathcal{L},\ u\neq 0,\ \tau\in\mathbb{R}_+. \tag{C.9}$$

(c) Robustness Margins

The closed-loop system has -6dB to +∞ multivariable gain margin and -60 to +60 degrees of multivariable phase margin at the input to H, i.e. the loop CΦH is robust.

(d) Closed-loop Stability

$$\|[I+C\Phi H]^{-1}\|_2 \leq 1 \tag{C.10}$$

and

$$\|C\Phi H[I+C\Phi H]^{-1}\|_2 \leq 2, \tag{C.11}$$

i.e. the closed-loop system is L_2-stable.

(e) L_2-domain inequality

$$\|[I+C\Phi H]u\|^2_{2,\tau} \geq \|u\|^2_{2,\tau} + \|\Xi^{1/2}S_x\Phi Hu\|^2_{2,\tau} \tag{C.12}$$

Proof The proof is carried out in a manner analogous to that of theorem 3.17. Let V be the function such that $V_x(S_x(x))=x$. Then

$$\frac{dV(S_x(x))}{dt} = x^T S_{xx}[f(x)+H(x)w] = x^T S_{xx} f(x) + x^T C^T w. \qquad \text{(C.13)}$$

and

$$\int_o^\tau [w^T w + S_x{}^T \Xi S_x] \, dt$$

$$= \int_o^\tau [w^T w + x^T C^T Cx - 2x^T S_{xx}(x) f(x)] \, dt$$

$$= \int_o^\tau \left\{ [w+Cx]^T [w+Cx] - 2w^T Cx - 2x^T S_{xx}(x) f(x) \right\} dt$$

$$= \int_o^\tau \left\{ [w+Cx]^T [w+Cx] - 2\dot{V} \right\} dt$$

$$= \int_o^\tau [w+Cx]^T [w+Cx] \, dt - [V(S_x(x(\tau)) - V(S_x(x(0))]. \qquad \text{(C.14)}$$

Since we start at $x(0)=0$ and $V \geq 0$, we have result (a). The others follow in a manner completely analogous to theorem 3.17. ∎

Remark These results are all completely dual to those of the optimal regulator (theorem 3.17). Thus it seems to be a good choice for an observer. However, we are missing one crucial property for the costate observer: incremental stability. From chapter 2 we know that this is a crucial property for a nondivergent estimator to have. Consider the observer equation (C.4). We know from the above results that, roughly, this equation is stable from $(u,y) \mapsto \hat{x}$. We simply need to upgrade to incremental stability in order to get nondivergence of the estimate. At this point no conditions are known that guarantee this property for the costate observer. It seems likely that this is linked to the incremental stability of solutions to the HJB eqaution, and that results should flow easily between the two, given their common form. There have been some results reported [67,68] on the incremental properties (called second-variation results) of the HJB equation, but they hold only along trajectories with zero input. We cannot say much about the incremental stability around other trajectories.

Thus it seems that the costate observer has many interesting properties, and seems to hold some potential as an observer which may be the "true" dual to the Hamilton-Jacobi-Bellman equation of optimal control theory.

APPENDIX D. Guaranteed Properties of the Optimal Filter

Theorem D.1 For the optimal deterministic filter, described by the equations

$$V_t = \frac{1}{2}[y-Cx]^T[y-Cx] - V_x{}^T f(x) - \frac{1}{2} V_x{}^T \Xi V_x \tag{D.1}$$

$$\hat{x}(t) = \underset{x}{\operatorname{argmin}}\ V(x,t) \tag{D.2}$$

where $V(x,0)\geq 0$, $V(0,0)=0$, and V is differentiable, let $\hat{y}=C\hat{x}$ and $v=y-\hat{y}$. Then the following "return difference condition" holds

$$\|v+\hat{y}\|_{2,\tau} \geq \|v\|_{2,\tau}\ ; \qquad \forall y\in\mathcal{L},\ \tau\in\mathbb{R}_+. \tag{D.3}$$

Proof Since V is differentiable, $V_x(\hat{x}(t),t)=0$, and thus we have

$$\frac{dV(\hat{x}(t),t)}{dt} = V_t(\hat{x}(t),t) + V_x(\hat{x}(t),t)\dot{\hat{x}}(t) = V_t(\hat{x}(t),t))$$

$$= \frac{1}{2}[y-\hat{y}]^T[y-\hat{y}] \tag{D.4}$$

and

$$\frac{dV(0,t)}{dt} = \frac{1}{2}y^Ty, \tag{D.5}$$

since $f(0)=0$. Thus we have

$$\|y\|^2_{2,\tau} = \|v+\hat{y}\|^2_{2,\tau} = V(0,\tau) - V(0,0) = V(0,\tau) \tag{D.6}$$

and

$$\|y-\hat{y}\|^2_{2,\tau} = \|v\|^2_{2,\tau} = V(\hat{x}(\tau),\tau) - V(\hat{x}(0),0) \tag{D.7}$$

and thus by the minimization property of $\hat{x}$, we have

$$V(\hat{x}(\tau),\tau) = \|v\|^2_{2,\tau} + V(\hat{x}(0),0) \leq V(0,\tau) = \|v+\hat{y}\|^2_{2,\tau}. \tag{D.8}$$

and since $V \geq 0$, we have the desired result. ∎

Remark This return difference property is very similar to the properties possessed by the extended Kalman filter, the costate observer, and the optimal regulator, as discussed in section 3.3.4, appendix C, and section 3.4.3, respectively. It is unclear exactly what this result implies for robustness margins of the optimal filter; it seems likely that it possesses the same margins as the above mentioned loops.

APPENDIX E. Factorization Ideas

In this section, we will discuss some of the factorization ideas [71] as they might relate to nonlinear systems. The Q-parameterization [27,28] we discussed under formal loop shaping was a basic form of factorization. In [27,28] the set of compensators stabilizing a given incrementally stable plant is parameterized (by Q). The factorization of [71] is more general and deals with possibly unstable linear plants. Suppose we have a plant given by $P=C\Phi B$ and we can find a stabilizing state feedback function G, i.e. so that

$$[\Phi^{-1}+BG]^{-1}B = \Phi B[I+G\Phi B]^{-1}, \tag{E.1}$$

and

$$[I+G\Phi B]^{-1} \tag{E.2}$$

are stable. Then we can find a right-factorization of P into two stable operators N, D:

$$P = ND^{-1}. \tag{E.3}$$

They are given by

$$N := C\Phi B[I+G\Phi B]^{-1} \tag{E.4}$$

$$D := [I+G\Phi B]^{-1}. \tag{E.5}$$

We would like to show that this is a right-coprime factorization, but this may require some additional assumptions. See [71] for the linear case and [32] for a nonlinear discussion.

We now consider the left-factorization. Suppose we have a nondivergent model-based filter with gain H for our plant P. In the standard filtering formulation that we have been using, let u,d=0. Then

$$N = [I+C\Phi H]^{-1}C\Phi B \tag{E.6}$$

$$D = [I+C\Phi H]^{-1} \tag{E.7}$$

are both stable. D is easy; N is stable because

$$\|[I+C\Phi H]^{-1}C\Phi Bw\|_T = \|y - C\hat{x}\|_T = \|C(x-\hat{x})\|_T \leq k\ |C|\ \|w\|_T. \tag{E.8}$$

We have

$$P = D^{-1}N, \tag{E.9}$$

and thus we have a left factorization (coprime?) for our plant. Perhaps it will be possible to use these factorizations to completely parameterize the set of compensators that will stabilize an arbitrary plant.

APPENDIX F. State Feedback Servos

In this section we will present a short description of the state feedback servo, in which a state feedback system is turned into a command following system.

Suppose that we have our plant

$$\dot{x} = f(x) + B\,u, \tag{F.1}$$

where we have partitioned the state vector so that

$$x = \begin{bmatrix} y \\ x_r \end{bmatrix} \tag{F.2}$$

and a stabilizing state feedback function $g(x) = Gx$. Now, suppose that we wish to use this feedback function G to create a command following system. We might try to use an input to our system (F.1)

$$u = g(x + Dr)$$

where r is a command reference input and

$$D = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

The system shown in figure F-1 shows this setup for the case where we can decompose G so that

$$g(x) = Gx = G_y y + G_r r. \tag{F.3}$$

The result that follows does not rely on this decomposition, but it does make the block diagram easier to compare with the linear case.

Theorem F.1 (State Feedback Servo Stability)
If the function g is Lipschitz in y, i.e.

$$|g(y_1,x_r)-g(y_2,x_r)| \leq M\ |y_1-y_2|, \tag{F.4}$$

then the closed-loop system with $u=g(x+Dr)$ is I/O stable.

Proof We have

$$\dot{x} = f(x) - B\ g(x+Dr) = f(x) - B\ g(x) + B[g(x)-g(x+Dr)] \tag{F.5}$$

and so

$$\|x\|_T \leq \|g(x)-g(x+Dr)\| \leq M\ \|r\|_T. \quad ∎ \tag{F.6}$$

Note that the Lipshitz condition means that g_y is Lipshitz if we use the decomposition (F.3).

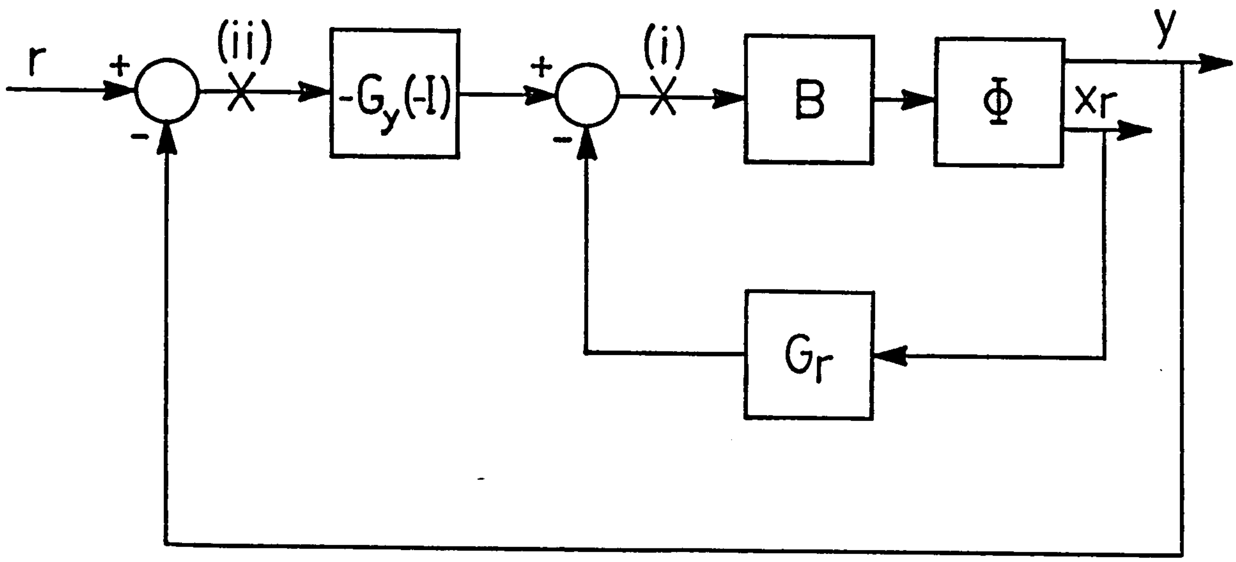


Figure F-1: State Feedback Servo

Remark The state feedback servo suffers from the same problems as its linear counterpart, the LQ servo. Note that we might have very good robustness and performance at the loop breaking point (i) in figure F-1, say if g came from an optimal regulator problem, but that the actual performance loop, point (ii), might be terrible. This concept is related to the discussion in section 4.6 where the two-step compensation methods were shown to have problems in terms of shaping the loops that are truly important.

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